SECTIONUM CONICARUM

ELEMENTA.

7.7.f

SECTIONUM CONICARUM
ELEMENTA
METHODO FACILLLIMA
DEMONSTRATA.

IN USUM JUVENTUTIS ACADEMICÆ.

AUTHORE L. TREVIGAR, A. M. k

AULÆ CLARENSIS SOCIO.

CANTABRIGIÆ.

TYPIS ACADEMICIS.

Proſtant apud N. PREVOST, & A. VANDENHOECK & apud
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MVSEVM
BRITAN
NICVM

HONORABILI VIRO

HENRICO BROMLEY

ARMIGERO,

IN COMITATU CANTABRIGIENSI

DOMINO LOCUM-TENENTI,

ET

CUSTODI ROTULORUM;

NECNON

IN COMITIIS PARLIAMENTARIIS

PRO EODEM COMITATU

EQUITI PROVINCIALI

HAS STUDIORUM SUORUM PRIMITIAS

HUMILLIME

DEDICAT

L. TREVIGAR.

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OPUSCULUM *hoc, Lector benevole, non tam de novo confectum, quam ex aliorum ſcriptis, præcipue autem ex illuſtriſſimi Marchionis de* L'HOSPITAL *tractatu excerptum ſenties; cum enim id mecum ſtatuiſſem, ut Sectionum Conicarum Elementa clariora prodirent, & ut Philoſophia* NEWTONIANA, *quantum ad Sectiones Conicas, intellectu facilior redderetur, neminem Autore noſtro potiorem duxi, qui huic propoſito ſatisfaceret; utpote qui nec longa & inutili demonſtrationum ſerie, nec difficili Harmonicalium proportione laboraret: ideoque Theoremata maximâ ex parte ab illo deſumpſi; quæque ad* NEWTONUM *intelligendum requiri & in Autore noſtro deſiderari videbantur, ea ab* APOLLONIO, MILNIO, *aliiſque magni nominis ſcriptoribus adjunxi, adeo ut quæ paſſim & membratim inter alios erant diffuſa, jam in unum quaſi corpus collecta videantur.*

Theoremata quædam de circulari Sectionum curvatura in libro quinto appoſui, eaque prorſus nova, apud nullos, quod ſcio, Sectionum Conicarum ſcriptores invenienda; His etiam accedit methodus proprietates omnes Parabolæ ex ſimilibus Ellipſeos vel Hyperbolæ proprietatibus derivandi; atque hic quàm latè pateat campus, quanta & quam utilia ex his principiis deduci poſſint, nemo, qui vel ſummis labris NEWTONIANAM *Philoſophiam guſtavit, non videt.*

Totum Autoris noſtri librum ſextum (de Sectionibus Conicis in Solido æſtimatis) prorſus omitto; etſi enim ibi omnes Sectionum

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Et ne brevitate nimis obſcurâ multo plus temporis in hiſce ſtudiis recolendis impenderetur, quam vel rei natura exigat, aut dignitas poſtulet, idcirco omnia quæ obſcurius & confuſius componi videbantur, ea meliori ordine diſponenda putavi, & quoad poſſem diſpoſui; non enim ut fidus interpres, verbum verbo curavi reddere, nec tamen priſtinam Autoris formam ita mutavi, ut novo prorſus cultu ornatus prodiret; at eum multo ſimpliciorem, uti ſpero, in medium profero, id ſcilicet ſtatuens, ut in Geometricâ vel mediocriter doctus, in Algebraicâ vel parum verſatus prima & præcipua Sectionum Conicarum Elementa ſine nimio ſudore percipere poſſit.

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LIBER PRIMUS
DE
PARABOLA

DEFINITIONES.

I.

Fig. 1.

SIT BC Regula in plano fixa; & GDO norma eâ lege poſita, ut DG unum ex ejus lateribus proximè applicetur ad inferiorem Regulæ partem: ſumatur filum FMO ejuſdem longitudinis cum DO, altero Normæ latere: ipſius autem DO extremitati O annectatur una fili extremitas, altera verò in quovis plani puncto F, ex eâdem Regulæ parte, quà Norma; jam ſi Normæ latus DG per Regulæ marginem BC moveri aut labi fingas, eodem autem tempore filum paxillo aliquo M continuo intendi, dum pars ejus MO lateri DO fixa, & quaſi agglutinata teneatur, Curva AMX, paxilli motu deſcripta, vocatur Parabola.

Conversa autem Normâ ſuper latus DO ad alteram partem puncti F, eodem modo deſcribitur altera ejuſdem Parabolæ pars AMZ, & linea XAZ erit tantum una eademque curva, quæ Parabola vocatur.

II.

Recta BC, inferior ſcilicet Regulæ fixæ pars, quæ & planum & Normam GDO tangit, appellatur Directrix.

III.

PUNCTUM F in plano fixum dicitur Parabolæ FOCUS, vel UMBILICUS.

IV.

SI a puncto F agatur recta FE ad Directricem BC normalis, Parabolæ occurrens in puncto A; recta AF indefinitè verſus F producta vocatur Parabolæ AXIS.

V.

RECTA p, quadrupla ipſius AF, dicitur eſſe axis PARAMETER, vel LATUS RECTUM.

VI.

OMNES rectæ MP, ex Parabolæ punctis ductæ ad axem normales, appellantur ORDINATIM APPLICATÆ, vel ORDINATÆ AD AXEM.

VII.

OMNES Rectæ MO, ex Parabolæ punctis ductæ, ad axem parallelæ, vocantur Parabolæ DIAMETRI.

VIII.

ILLA Axis vel Diametri cujuſcunque pars, inter verticem & Ordinatam intercepta, vocatur illius Axis vel diametri ABSCISSA; adeo ut AP, AP, dicantur ABSCISSÆ.

IX. RECTA.

IX.

RECTA, quæ Parabolæ in uno puncto occurrit, utrinque autem producta extra cadit, dicitur TANGENS in iſto Parabolæ puncto.

COROLLARIUM. I.

1. CONSTAT ex Parabolæ definitione, ſi a puncto quovis M ducatur ad focum F recta MF, & ad Directricem BC normalis MD, rectas MF, MD, ſibi invicem ſemper æquari.

Nam * $OM + MF = OM + MD$ * Def. 1.

Ablato communi OM, erit $MF = MD$

COROLLARIUM. II.

2. UNDE ſi ducatur recta quæcunque KK directrici BC parallela, & ex quovis Parabolæ puncto M recta MK ipſi KK normalis, & ad Focum F recta MF; KD differentia vel ſumma rectarum MF, MK, eadem ſemper * erit; differentia ſcilicet, quando M infra K; & ſumma, quando M ſupra K inveniatur. * 34 Elem. 1.

COROLLARIUM. III.

3. CONSTAT lineam FE bifariam ſecari in puncto A; cadente enim puncto M in A, neceſſe eſt recta MF cadat ſuper rectam AF, recta autem MD ſuper AE (nam punctum F eſt fixum) ſed * MF ſemper æqualis eſt ipſi MD; unde AF ipſi AE æqualis erit. * Art. 1.

COROLLARIUM. IV.

4. HINC patet Methodus deſcribendi Parabolam XAZ, ex dato axe AP (cujus vertex, A,) & ex datâ parametro p; ſumantur enim ſuper axem AP, ex utraque parte puncti A, rectæ AF, AE, ſingulæ æquales quartæ parti Parametri p datæ, & per punctum E ducatur recta indefinita BC ipſi FE normalis; Deinde ſi ponatur inferior Regulæ pars ſuper rectam BC pro Directrice habitam, & applicetur norma GDO, ita & filum FMO lateri OD æquale, cujus altera extremitas ad punctum O, altera ad F ſit fixa, & deſcribatur Parabola XAZ, ut in prima definitione, liquet hanc fore Parabolam quæſitam:

CONSTAT etiam, quo longius fuerit normæ latus
* Def. 1. OD, hoc eſt, filum OMF, (ipſi OD ſemper * æquale)
eò majorem eſſe Parabolæ deſcriptæ portionem: adeo ut pro libitu augeri vel diminui poſſit curva Parabolica, ſi & normæ latus OD, & filum OMF æqualiter augeantur vel diminuantur.

COROLLARIUM. V.

5. SI ex quovis Parabolæ puncto M agatur ad axem Ordinata quævis MP, & ad focum F recta MF, erit $MF = AP + AF$

* Art. 1. Nam * $MF = MD = AP + AE$
* Art. 3. Sed * $AE = AF$
Unde $MF = AP + AF$

PRO-

PROPOSITIO. I.

THEOREMA.

6. *Quadratum Ordinatæ cujuſcunque* MP *ad axem* AP, *æquatur rectangulo ſub parametro p, & Abſciſſâ* AP. Fig. 1.

Dico $\overline{MP}^2 = p \times AP$

Sit $AF = m$. $AP = x$. $PM = y$.

Unde *$MF = m + x$. Et $PF = m - x$, vel $x - m$; ſcilicet * Art. 5. $m - x$, quando punctum P eſt ſupra, & $x - m$, quando P eſt infra focum F.

Jam vero,

* $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$ in utroque caſu. * 47. Elem. 1.

Hoc eſt

$$mm + 2mx + xx = yy + mm - 2mx + xx$$

Unde $yy = 4mx$

Sed * $4m = p$ * Def. 5.

Ergo $yy = px$. Hoc eſt, $\overline{MP}^2 = p \times AP$.

COROLLARIUM PRIMUM ET FUNDAMENTALE.

7. MANIFESTUM eſt igitur, ſi parameter ipſius axis AP dicatur p; ſingulæ axis partes AP, x; Ordinatæ ejus quæcunque PM, y; ſemper fore $yy = px$. Et cum hæc proprietas omnibus Parabolæ punctis æquè conveniat, atque eorum ſitum quoad axem conſtanter definiat, ſequitur æquationem $yy = px$ Parabolæ naturam, quoad ejus axem, perfectè & ad amuſſim exprimere & determinare. Fig. 2.

COROL.-

COROLLARIUM. II.

Fig. 2. 8. Si ducantur duæ Ordinatæ quæcunque MP, NQ, ad axem AP; quadrata earum erunt ad ſe invicem, ut axis Abſciſſæ AP, AQ.

Dico $\overline{MP}^2 : \overline{NQ}^2 :: AP : AQ.$

* Art. 6. Nam* $\overline{MP}^2 = p \times AP.$ & $\overline{NQ}^2 = p \times AQ$

Unde $\overline{MP}^2 : \overline{NQ}^2 :: p \times AP : p \times AQ.$

* 1. Elem. 6. Sed * $p \times AP : p \times AQ :: AP : AQ$

Ergo $\overline{MP}^2 : \overline{NQ}^2 :: AP : AQ.$

COROLLARIUM. III.

9. Hinc ſi Ordinatim applicatæ quotlibet æqualibus ab invicem diſtantiis ſumantur, erunt earum quadrata in continua progreſſione Arithmetica: hoc eſt, ut numeri 1, 2, 3, 4, 5 *&c.* ac proinde ipſæ ordinatæ ut 1, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$ *&c.*

COROLLARIUM. IV.

10. Si per punctum quodvis P in axe AP agatur recta MPM axis Ordinatis parallela, ea Parabolæ occurret in duobus punctis M & M a puncto P æquè remotis; rectæ enim MP, MP, cum ponantur eſſe ad axis Ordinatas parallelæ, Parabolæ occurrunt (ei enim occurrunt Ordinatæ ipſæ) & ad axem *erunt normales, ideoque ad Axem Ordinatæ. Unde ſingula quadrata * Def. 6.

$\overline{MP}^2$, $\overline{MP}^2$, eidem rectangulo px æquantur, & $MP = MP$; hoc eſt, puncta, M & M æqualiter diſtant a puncto P.

COROL-

COROLLARIUM. V.

11. Cum * ſit $yy=px$. vel $\overline{MP}^2=p\times AP$, liquet, quo major fuerit AP vel x, eo majorem fore PM vel y, ex utraque axis parte ſumptam: adeo ut ſi AP in infinitum producatur, PM infinitè etiam augebitur. Et è contrario, quo minor fuerit AP, eò minor erit Ordinata PM; ideoque ſi AP fiat infinitè parva, vel evaneſcat, PM ex utraque axis parte erit quoque infinitè parva vel evaneſcet: hoc eſt, ſi punctum P cadat ſuper punctum A, puncta M & M ſibi mutuo occurrent in P. unde patet,

* Art. 7.

1*mo*. RECTAM LL per axis verticem A ductam, & ad Ordinatas parallelam, eſſe tangentem in puncto A.

2*do*. SINGULA Parabolæ puncta eo longius ab axe ejus AP diſtare, quo remotiora ſint a vertice A; ideoque recta LM, axi AP parallela, Parabolæ non niſi in uno puncto occurrit, diſtantia enim ejus ab axe eadem ſemper manet, a Parabolâ autem continùo augetur.

COROLLARIUM. VI.

12. SI ex quovis Parabolæ puncto M agatur recta ML axi AP parallela, rectæ autem AL Ordinatis parallelæ occurrens in L, patet eſſe * $AL=PM=y$. Ob eandem rationem $ML=AP=x=\frac{yy}{p}$ (nam * $px=yy$ & dividendo per p, erit $x=\frac{yy}{p}$.) Unde ſequitur rectas ML, ML, ex utraque axis parte ſumptas, tum ſibi invicem æquari, cum puncta L & L æqualiter diſtant a puncto A; nam ex iſtâ Hypotheſi $LA=LA$, hoc eſt $PM=PM$, ac proinde $\frac{yy}{p}=\frac{yy}{p}$ (nam p quantitas eadem eſt & conſtans) ergo $ML=ML$.

* 34. Elem. 1.

* Art. 7.

COROL-

COROLLARIUM. VII.

13. Si recta MM, ad Parabolam utrinque terminata, bifariam ſecetur ab axe AP, ea tangenti LL parallela erit; ductis enim ML, ML, axi AP parallelis, liquet rectam LL bifariam ſecari in puncto A, quia MM biſecatur in
* Art. 12. P: unde *ML, ML, ſibi invicem æquantur; ideoque
* 33. El. 1. *erit MM ipſi LL parallela. Eſt etiam MM ordinata
* Art. 11. ad axem, recta enim LL *parallela eſt Ordinatis ad Axem: ſed & rectæ MM eſt parallela, unde & ipſa MM erit Ordinata.

COROLLARIUM. VIII.

14. Cum omnes perpendiculares MPM ab axe AP bifariam ſecentur, conſtat Parabolam ipſam ab axe dividi in duas partes æquales, ſimiles, & ſimiliter poſitas; ſi enim plani Parabolici pars una in alteram ſuperimponatur, hoc eſt, ſi omnes Ordinatæ MP ponantur ſuper omnes Ordinatas PM, hæ illis perfectè congruent, ac proinde tota Parabolæ portio ex una axis parte coincidet cum tota parabolæ portione ex altera axis parte, cum portiones iſtæ ex infinito Ordinatarum numero componantur & conſtituantur.

LEMMA. I.

Fig. 3. 15. Sint ABC, DEF *duo triangula, quorum latera ſibi mutuo parallela fuerint; videlicet* AB *ipſi* DE, AC *ipſi* DF, BC *ipſi* EF, *dico hæc triangula eſſe ſimilia.*

* 10. Elem. 11. Ob latera AB, BC ipſis DE, FE parallela, *erit angulus ABC=DEF: ob eandem rationem angulus BAC
=EDF, ita & angulus ACB=DEF. unde triangula
*4. Elem. 6. *erunt ſimilia.

PRO-

PROPOSITIO II.

Theorema.

16. Si *per axis* AP *verticem* A *ducatur in utrovis an-* Fig. 4. *gulorum* PAL, PAL, *ſub axe* AP *&* *rectâ* LL *ordinatis parallela contentorum, recta quæcunque* AM; *dico hanc Parabolæ* MAM *iterum occurrere in alio puncto* M.

Sumpta enim ſuper AL, ex utravis parte puncti A, rectâ AG axis parametro p æquali; actâque GF axi AP parallelâ, quæ ipſi AM (productæ ſi opus) in puncto F occurrat; capiatur ſuper rectam AL (ex eadem axis parte, qua AM) pars AL ipſi GF æqualis; & fiat LM axi AP parallela; dico punctum M, ubi LM ipſi AM occurrit, eſſe ad Parabolam.

Sit enim MP ipſi AL parallela, & triangula FGA, APM erunt * ſimilia; ac proinde ſi fiat $PM=y$, $AP=x$, * Art. 15. $GA=p$. erit FG, vel AL, vel $PM : GA : : AP : PM$, Hoc eſt, $y : p : : x : y$. Unde * $yy=px$; PM igitur erit * 16. El. 6. Ordinata * ad axem, adeoque punctum M eſt ad Pa- * Art. 7. rabolam.

Corollarium I.

17. Hinc ex datis Parabolæ axe AP, & ejus parametro p, ductâque per verticem A rectâ AM in utrovis angulorum PAL, PAL, facilè innoteſcit punctum M, ubi recta AM Parabolæ occurrit: Ducatur ſcilicet recta AG, parametro datæ p æqualis, & recta GF, axi AP parallela; Productâ GF, uſque dum ipſi AM occurrat in F, fiat AL=GF, & LM ipſi GF parallela; liquet igitur punctum M, ubi recta LM ipſi AM occurrit, * eſſe punctum quæſitum. * Art. 16.

COROLLARIUM II.

18. MANIFESTUM eſt, nullam rectam præter LAL Parabolam tangere poſſe in puncto A; ea enim eſt unica, quæ, per verticem A producta, tota cadat extra Parabolam; quamcunque enim duxeris ab LAL diverſam, ea in angulo PAL neceſſario cadet, ac proinde Parabolæ
* Art. 16. in alio puncto præter A * occurrit.

DEFINITIONES.

X.

Fig. 5 & 6. SI per quodvis Parabolæ punctum M agantur Diameter MO, Ordinata MP ad axem AP, & recta MT, quæ ab axe AP, ultra verticem ejus producto, auferat partem AT ipſi AP æqualem; omnes rectæ NO, ductæ ex Parabolæ punctis quibuſcunque ipſi MT parallelæ, & ad Diametrum MO terminatæ, vocantur ORDINATÆ, vel ORDINATIM APPLICATÆ AD DIAMETRUM MO.

XI.

SI fiat recta q tertia proportionalis ad ipſas AT, MT, iſta recta q dicitur PARAMETER diametri MO.

COROLLARIUM I.

19. FIAT AP, vel AT $= x$. & erit $\overline{MT}^2 = qx$.

* Def. 11. Nam * AT : MT : : MT : q.

Hoc eſt x : MT : : MT : q.

* 17. El. 6. Ergo * $\overline{MT}^2 = qx$.

COROL-

COROLLARIUM II.

20. PARAMETER q diametri cujuſcunque MO, excedit parametrum p axis AP quantitate $4x$, quadrupla axis AP (x).

Nam $*\overline{MT}^2 = \overline{MP}^2 + \overline{PT}^2$. * 47. El. 1.
Hoc eſt $*qx = px + 4xx$. * Art. 19.
Unde $q = p + 4x$.

COROLLARIUM III.

21. PARAMETER q diametri cujuſcunque MO, æquatur quadruplæ rectæ MF, ab illius diametri origine M ad focum F ductæ.

Scilicet $q = 4$MF.
Nam $*q = p + 4x$, * Art. 20.
Sed $*p = 4$AF, & $4x = 4$AP. * Def. 5.
Ergo $q = 4$AF $+ 4$AP,
Sed $*4$MF $= 4$AF $+ 4$AP, * Art. 5.
Unde $q = 4$MF.

PROPOSITIO III.

THEOREMA.

22. QUADRATUM *Ordinatæ cujuſvis* ON *ad diametrum* MO, *æquatur rectangulo ſub parametro q, & abſciſſâ* MO.

Dico $\overline{ON}^2 = q \times$ MO.

DUCTA ad axem AP ordinata NQ diametro MO in puncto R occurrat; & ſit OH ipſi MP parallela. Fig. 5. & 6.

Fiat AP vel AT $= x$. PM vel RQ $= y$. OR vel HQ $= a$. MO vel PH $= b$.

Jam

* Art. 15. Jam triangula TPM, ORN, ſunt * ſimilia.

Unde TP : PM : : OR : RN.

Hoc eſt, $2x : y :: a : \frac{ay}{2x} = RN$,

His poſitis, erit in Fig. 5ta.

$NQ = RQ - RN = y - \frac{ay}{2x} = \frac{2xy - ay}{2x}$,

Vel $NQ = RN - RQ = \frac{ay}{2x} - y = \frac{ay - 2xy}{2x}$,

In utroque caſu, $\overline{NQ}^2 = yy - \frac{ayy}{x} + \frac{aayy}{4xx}$.

Porro in Fig. 6ta.

$NQ = RQ + RN = y + \frac{ay}{2x}$,

Et $\overline{NQ}^2 = yy + \frac{ayy}{x} + \frac{aayy}{4xx}$,

Unde in figuris 5ta & 6ta erit,

$\overline{NQ}^2 = yy \mp \frac{ayy}{x} + \frac{aayy}{4xx}$.

Rurſus in Fig. 5ta erit,

$AQ = AH - HQ = AP + PH - HQ = x + b - a$.

In Fig. 6ta.

$AQ = AH + HQ = AP + PH + HQ = x + b + a$,

Ideoque in figuris 5ta & 6ta, erit $AQ = x + b \mp a$,

* Art. 8. Jam vero, *AP : AQ : : $\overline{PM}^2 : \overline{QN}^2$:

Hoc eſt, $x : x + b \mp a :: yy : yy + \frac{byy}{x} \mp \frac{ayy}{x}$,

Unde $\overline{QN}^2 = yy + \frac{byy}{x} \mp \frac{ayy}{x}$;

Sed $\overline{QN}^2 = yy \mp \frac{ayy}{x} + \frac{aayy}{4xx}$,

Ergo $yy \mp \frac{ayy}{x} + \frac{aayy}{4xx} = yy + \frac{byy}{x} \mp \frac{ayy}{x}$;

Auferendo utrinque $yy \mp \frac{ayy}{x}$, dividendo per yy, & ducendo in $4xx$, erit $aa = 4bx$,

Scilicet $\overline{OR}^2 = 4bx$;

Sed triangula MPT, NRO ſunt *ſimilia, * Art. 15.

ac proinde $\overline{PT}^2 : \overline{OR}^2 :: \overline{MT}^2 : \overline{ON}^2$,

Hoc eſt, $4xx : 4bx :: {}^{*}\, qx : \frac{4bqxx}{4xx} = bq$. * Art. 19.

Ergo $\overline{ON}^2 = bq = q \times MO$.

COROLLARIUM I.

23 MANIFESTUM eſt, quæcunque demonſtrata fuerint in propoſitione prima de axe AP, locum habere apud omnes diametros MO, earum Ordinatas ON, & parametrum *q*. Et cum articuli 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, a prima propoſitione deducantur, verique ſint, utrum anguli APM recti fuerint, necne; ſequitur, ſi fingamus rectam AP non eſſe axem, ſed diametrum quamcunque, cujus Ordinatæ ſint PM, NQ, & parameter *p*, hos articulos adhuc veros fore, eorum enim demonſtratio prorſus eadem erit.

COROL-

COROLLARIUM II.

Fig. 5 & 6. 24. Cum articuli 11 & 18 æqualiter obtineant, ſive AP ſit axis, ſive Diameter quæcunque ut MO, conſtat rectam MT, diametri Ordinatis parallelam, eſſe tangentem in puncto M, nullamque aliam rectam in illo puncto Parabolam tangere poſſe.

Unde patet, unam ſolummodo tangentem duci poſſe per datum Parabolæ punctum.

COROLLARIUM III.

25. Hinc, ſi per quodvis Parabolæ punctum M agatur ad axem ordinata quævis MP, & recta MT, quæ ab axe, ultra verticem producto, auferat partem AT ipſi AP æqualem, recta MT erit contingens in puncto M; nam MT parallela *est Ordinatis ad Diametrum, ac proinde Parabolam *tangit in puncto M.

* Def. 10.
* Art. 24.

Et vice verſâ, ſi recta MT Parabolam in puncto M tangat, & ad axem ducatur Ordinata MP, partes AT, AP, erunt æquales; non enim: ſed ſi fieri poteſt, non ſit AT ipſi AP æqualis; liquet igitur rectam aliquam duci poſſe per punctum M, quæ auferat partem, ipſi AP æqualem; & hæc erit tangens: unde duæ rectæ Parabolam tangunt in eodem puncto M, * quod fieri nequit.

* Art. 24.

COROLLARIUM IV.

Fig. 7. 26. Si in definitionibus 10 & 11, & in ultima hac propoſitione, ponamus rectam AP non eſſe axem, ſed diametrum quamcunque, cujus Ordinatæ ſint PM, QN, adhuc conſtabit hujus propoſitionis veritas, cum eadem prorſus ſit demonſtratio, ut ex ſola figuræ ſeptimæ inſpectione patet; ex ſimilibus enim triangulis TPM, ORN, eadem elicitur proportio, ac cum AP pro axe habebatur.

UNDE ſi ex Parabolæ quovis puncto M agatur ad Diametrum AP ordinata MP, ita & recta quædam MT, eandem AP ita ſecans in T, ut ſit AT=AP, erit MT tangens in puncto M, & vice verſâ.

LIQUET etiam, Diametrum MO (ex hac Hypotheſi) pro axe ipſo haberi poſſe; ideoque axis pro diametro habendus, quæ cum Ordinatis ejus faciat rectos angulos.

PROPOSITIO IV.

THEOREMA.

27. DUCTIS *ad Diametrum quamcunque* AP *Semi-ordinatis* MP, NQ; *dico parametrum p eſſe ad ſummam Semiordinatarum* MP, NQ, *ut differentia illarum ad differentiam earundem abſciſſarum* AP, AQ. Fig. 8.

Scilicet $p : MP + NQ :: NO : MO$.

Nam * $AQ = \frac{\overline{NQ}^2}{p}$. Et $AP = \frac{\overline{MP}^2}{p}$, * Art. 6. & 22.

Ergo $AQ - AP = \frac{\overline{NQ}^2 - \overline{MP}^2}{p}$;

Unde $\overline{AQ - AP} \times p = \overline{NQ}^2 - \overline{MP}^2$.

Sed $\overline{NQ}^2 - \overline{MP}^2 = \overline{NQ + MP} \times \overline{NQ - MP}$,

Ergo $\overline{AQ - AP} \times p = \overline{NQ + MP} \times \overline{NQ - MP}$,

Vel, * $p : NQ + MP :: NO : MO$. * 16. El. 6.

PRO-

PROPOSITIO V.

THEOREMA.

Fig. 2. 28. SI *per quodvis Parabolæ punctum* M *ducatur ad ad axem Ordinata* MP, *& ad tangentem, per punctum* M *ductam, demittatur normalis quædam* MG; *dico axis partem* PG *dimidiæ parametri p ſemper eſſe æqualem.*

Hoc eſt, $PG = \frac{1}{2}p$.

* Def. 6. Quoniam MP ipſi TG normalis *ſit, erunt triangula TPM, GPM *ſimilia, ac proinde
*8. Elem. 6.

$$TP : PM :: PM : PG$$

Hoc eſt, $2x : y :: y : \frac{yy}{2x}$ unde $PG = \frac{yy}{2x}$;

* Art. 7. Sed * $yy = px$. Ergo $PG = \frac{px}{2x} = \frac{1}{2}p$.

PROPOSITIO VI.

THEOREMA.

Fig. 9. 29. SI *ab axis* AP *vertice* A *erigatur uſque ad quamlibet contingentem* MT *perpendicularis* AK; *& a puncto* K *erecta ad contingentem perpendicularis* KF *ſecet axem in* F; *erit ubique* AF *æqualis quartæ parti Parametri axis.*

Scilicet $AF = \frac{1}{4}p$.

* Art. 25. A tactu M ordinatâ MP ad Axem AP, erit *$AP = AT$.

* 2. El. 6. Unde * $AK = \frac{1}{2}MP$, & $\overline{AK}^2 = \frac{1}{4}\overline{MP}^2$.

* 8. El. 6. Sed * $\overline{AK}^2 = AF \times AT$,

Et

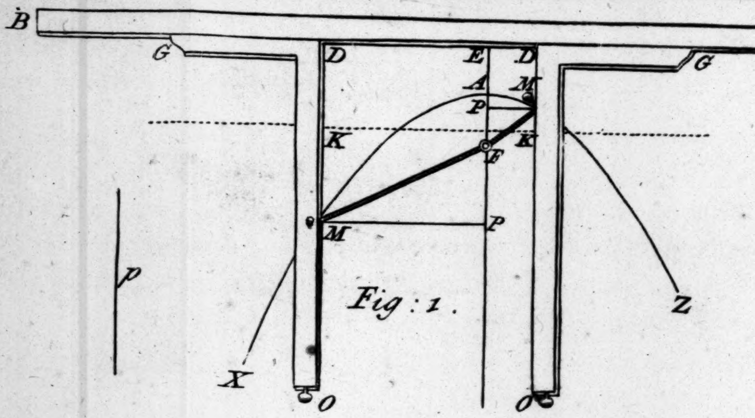


Fig: 1.

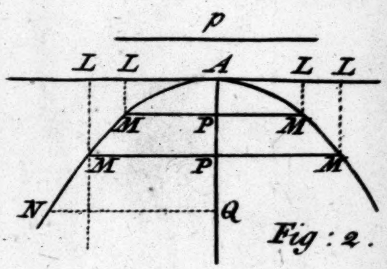


Fig: 2.

Fig: 3.

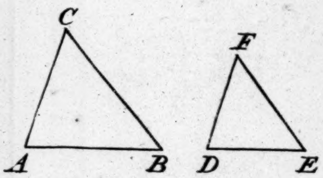
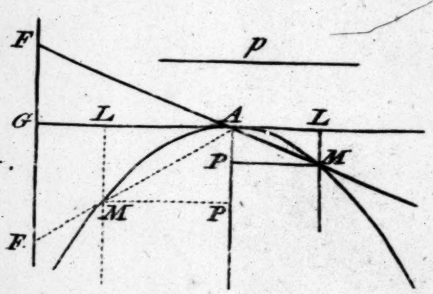


Fig: 4.



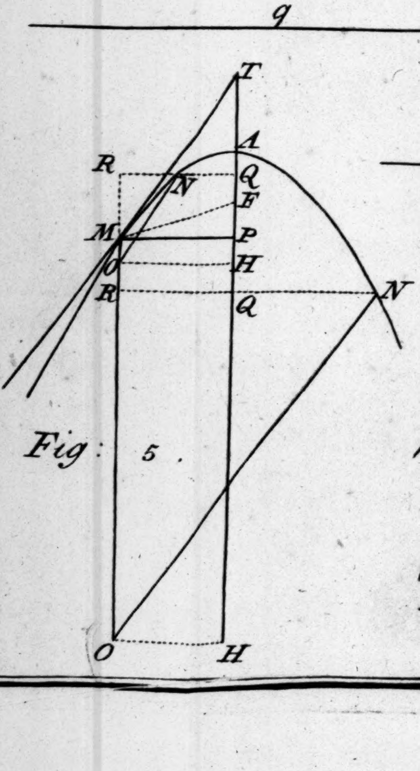


Fig: 5.

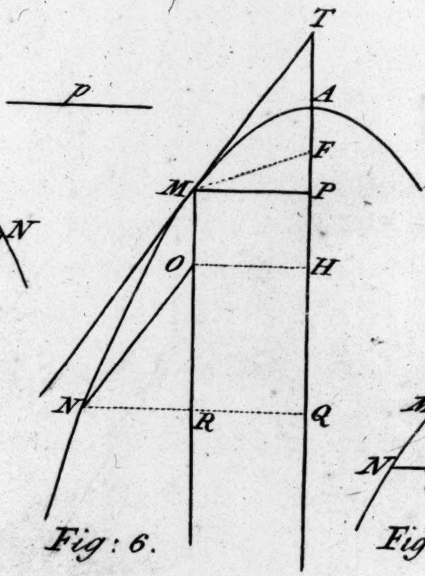


Fig: 6.

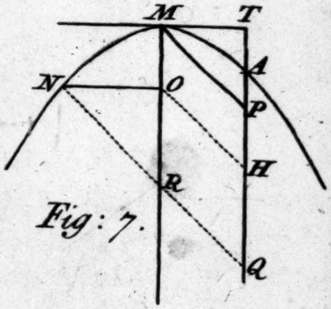


Fig: 7.

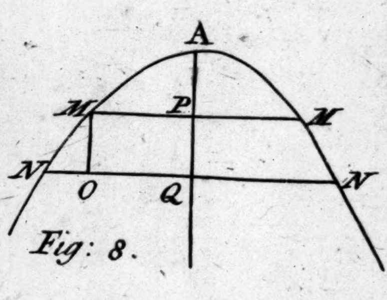


Fig: 8.

Et *$\frac{1}{4}\overline{MP}^2 = AP \times \frac{1}{4}p = AT \times \frac{1}{4}p$. * Art. 6.
Ergo $AF \times AT = AT \times \frac{1}{4}p$,
Hoc eſt, $AF = \frac{1}{4}p$.

COROLLARIUM I.

30. HINC conſtat, * punctum F eſſe Parabolæ FOCUM. * Def. 3. & 5.

COROLLARIUM II.

31. RECTA HF, per focum ducta, & ad axem Ordinata, æquatur ſemiparametro p; nam * $AF = \frac{1}{4}p$, unde * Art. 29.
*erit $\overline{HF}^2 = \frac{1}{4}p \times p = \frac{1}{4}pp$. Ergo $HF = \frac{1}{2}p$. * Art. 6.

PROPOSITIO VII.

THEOREMA.

32. SI *per quodvis Parabolæ punctum* M, *agatur ad Focum* F *recta* MF, *& ex eodem puncto diameter quævis* MO, *& tangens* TMS; *dico angulum* FMT, *ab una tangentis parte* TM, *& recta* MF *factum, æquari angulo* OMS, *ſub alterâ tangentis parte* MS, *& Diametro* OM *comprehenſo*.

Scilicet $FMT = OMS$.

Producatur Axis AP, donec tangenti TMS occurrat in T, & ſit MP ordinata ad axem: unde erit * $TA = AP$, * Art. 25.
Addatur utrinque AF,
Et erit $TA + AF = AP + AF$,
Sed * $MF = AP + AF$, * Art. 5.
Unde $TF = MF$,

* 5. El. 1. Et ang. * FMT = FTM;
* 27. El. 1. Sed ang. * FTM = OMS,
Ergo FMT = OMS.

COROLLARIUM I.

33. HINC ſi a Foco F demittatur normalis FK ad * 26. El. 1. tangentem MT, illa tangentem bifariam * ſecabit in puncto K.

COROLLARIUM II.

34. UNDE MF × AF = $\overline{FK}^2$; nam ob triangulum MFT Iſoceles, & demiſſam perpendicularem FK, erit * 4. El. 6. angulus TFK = KFM, & triangulum AFK * ſimile triangulo KFM; ergo erit AF : FK : : FK : MF, ideoque MF × AF = $\overline{FK}^2$.

PROPOSITIO VIII.

PROBLEMA.

Fig. 10. 35. DATIS *Diametro, & tangente per verticem ejus ductâ, ita & parametro ejus; Parabolam ipſam motu continuo deſcribere.*

SI diameter data eſſet axis, facile deſcribi poſſet Parabola ſecundum articulum quartum; ſin minus, ſit MO diameter data, & TMS tangens per verticem ejus ducta.

SUPER Diametrum ultra verticem M productam, ſumatur pars MD æqualis quartæ parti Parametri ejus datæ: per punctum D agatur DE ipſi MD normalis, ducaturque MF ita, ut angulus FMT æqualis ſit angulo OMS; factâ deinde MF ipſi MD æquali, deſcribatur, ut in primâ definitione, Parabola, cujus directrix ſit DE, & Focus F; dico hanc fore Parabolam quæſitam.

RECTA enim MO, cum directrici DE normalis ſit,
axi parallela erit, ideoque etiam *diameter: Recta autem * Def. 7.
TMS Parabolam *tangit in puncto M; & Parameter * Art. 32.
diametri MO *eſt quadrupla ipſius MF. * Art. 21.

PROPOSITIO IX.

PROBLEMA.

36. DATIS *Diametro* AP, *Parametro ejus, & tangente* AL, *per iſtius diametri verticem* A *ductâ; Parabolam per plurima puncta deſcribere.* Fig. 11.

DUCTA, per tangentis AL punctum quodvis L, rectâ indefinita LE diametro AP parallelâ; ſumantur ſuper LE & ſuper diametrum AP ultra verticem ejus productam, partes LE, EE, EE, *&c.* AF, FF, FF, *&c.* ſibi mutuo æquales, cujuſvis autem magnitudinis. Notetur ſuper LE punctum M, ita ut LM ſit tertia proportionalis ad parametrum Diametri AP datam, & ad partem AL tangentis datæ. Ductis igitur per puncta A & M rectis AE, AE, AE, *&c.* MF, MF, MF, *&c.* dico puncta interſectionis, N, N, N, *&c.* eſſe ad Parabolam quæſitam.

ACTIS enim per punctum notatum M & per punctum aliquod inventum N, rectis MP, NQ, tangenti AL parallelis; fiat AP $= x$. PM vel AL $= y$. AQ $= v$. NQ $= z$.

* Art. 15. Jam vero Triangula NQA, ALE *ſunt ſimilia; ita
* 2 Elem. 6. & *MPF, NQF. Unde erit,

NQ : QA :: AL : LE,

Hoc eſt, $z : v :: y : \frac{vy}{z}$, Unde LE vel AF $= \frac{vy}{z}$.

Rurſus MP : PF vel PA + AF :: NQ : QF vel QA + AF,

Hoc eſt, $y : x + \frac{vy}{z} :: z : v + \frac{vy}{z}$;

* 16. El. 6. Ergo * $vy + \frac{vyy}{z} = xz + vy$; deleto utrinque vy, &
erit $\frac{vyy}{z} = xz$, vel $vyy = xzz$.

* 16. El. 6. Unde * $x : v :: yy : zz$.

Hoc eſt, AP : AQ :: $\overline{MP}^2 : \overline{NQ}^2$.

*Conſtruct. Jam vero quadratum ex AL vel PM *æquatur rectangulo ſub diametri parte AP, & Parametro ejus; unde
* Art. 22. *erit PM ordinata ad diametrum AP, ideoque etiam QN; Punctum igitur N erit in Parabola ex alterutra parte diametri AP. Ideoque ad invenienda puncta ex altera parte iſtius diametri, capiantur, ſuper rectas indefinitas LE, AF, partes æquales LE, EE, *&c.* AF, FF, *&c.* ex altera parte punctorum L & A.

LIBER

LIBER SECUNDUS
DE
ELIPSI
DEFINITIONES.

I.

SUMATUR filum FM*f*; & in plano quovis affigantur ejus extremitates ad puncta F, *f*, quorum diſtantia ab invicem longitudine fili minor ſit; deinde ope paxilli M continuo tenſum teneatur filum, & eodem tempore circumagatur paxillus circa puncta F, *f*, donec in orbem rediens ad idem punctum, unde digreſſus eſt, iterum revertatur. Linea autem paxilli motu deſcripta, vocatur ELLIPSIS. Fig. 12, & 13.

II.

PUNCTA fixa F, *f*, vocantur Ellipſeos FOCI, ſeu UMBILICI.

III.

RECTA A*a* per focos ducta, & ab Ellipſi utrinque terminata, dicitur PRIMUS vel MAJOR AXIS, vel LATUS TRANSVERSUM, vel AXIS PRINCIPALIS, vel TRANSVERSUS.

IV.

PUNCTUM C, ubi primus axis A*a* bifariam ſecatur, appellatur Ellipſeos CENTRUM.

RECTA

V.

Recta B*b* per Ellipſeos centrum ducta ad primum axem A*a* perpendicularis, & ab Ellipſi utrinque terminata, dicitur **Secundus**, vel **Minor Axis**.

VI.

Axes A*a*, B*b*, ſimul ſumpti appellantur **Conjugati**; adeo ut primus axis A*a* dicatur eſſe ſecundo B*b* **Conjugatus**; & viciſſim ſecundus B*b* primo axi A*a* **Conjugatus**.

VII.

Rectæ MP, MK, per Ellipſeos puncta, ut M, ductæ, uni axium parallelæ, & ab altero terminatæ, dicuntur eſſe ad hunc alterum **Ordinatæ**; adeo ut MP ſit ad Axem A*a* **Ordinata**, & MK ad Axem B*b*.

VIII.

Tertia proportionalis ad duos axes, vocatur **Parameter** primi in proportione termini: ſi igitur fiat, ut axis A*a* ad axem B*b*, ita axis *Bb* ad tertiam *p*, tertia illa *p* erit axis A*a* **Parameter**.

IX.

Omnes rectæ per Centrum C ductæ, & ab Ellipſi utrinque terminatæ, **Diametri** appellantur.

X.

Recta, quæ Ellipſi in uno tantum puncto occurrit, utrinque autem producta extra cadit, TANGENS eſt in illo Ellipſeos puncto.

XI.

ILLA axis vel diametri cujuſvis pars, inter Ordinatam & verticem comprehenſa, vocatur ABSCISSA.

SCHOLIUM.

37. SI puncta F, *f*, in unum coeant ad centrum C, Fig. 14. conſtat Ellipſin mutatam iri in circulum, cujus radius didio fili CMC æqualis ſit; adeo ut circulus pro Ellipſi haberi poſſit, in quâ focorum diſtantia evaneſcat; quæcunque igitur in ſequentibus de Ellipſi fuerint demonſtrata, etſi maxima fuerit focorum ab invicem diſtantia, de Circulo tamen æquè vera ſunt, & æqualiter obtinent, ſi modo ponamus diſtantiam hanc evaneſcere vel nullam fieri.

COROLLARIUM I.

38. CONSAT ex primâ definitione, ſi ex quovis Ellipſeos puncto M agantur ad focos F, *f*, rectæ MF, M*f*, Fig. 12. & 13. earum ſummam ſemper eſſe eandem; utpote quæ fili ipſius longitudini ſemper æqualis ſit.

COROLLARIUM II.

39. CADENTE puncto M in A, neceſſe eſt recta Fig. 12. MF cadat ſuper rectam AF, ita & recta M*f* ſuper A*f*. Ob eandem rationem, puncto M in *a* cadente, recta Fig. 13. MF cadit ſuper *a*F, & M*f* ſuper *af*. Unde patet eſſe, AF + A*f* = MF + M*f*.

Hoc eſt, $2AF + Ff = MF + Mf$,

Sed $2af + Ff = Mf + MF$ ob eandem rationem.

Ergo $2AF + Ff = 2af + Ff$. Unde $AF = af$.

Hinc ſequitur 1*mo*, ſummam rectarum, MF, Mf, axi majori Aa ſemper æquari.

Nam $MF + Mf = AF + Af$,

Sed $AF = af$,

Ergo $MF + Mf = Af + af = Aa$.

2*do*. Liquet focorum diſtantiam Ff biſecari a Centro C,

* Def. 4. Nam * $CA = Ca$; & $AF = af$,

Unde $CA - AF = Ca - af$ hoc eſt, $CF = Cf$.

Corollarium III.

Fig. 12. 40. Si a termino B axis ſecundi Bb ducantur ad duos

* Def. 5. focos F, f, rectæ BF, Bf, patet triangula * rectangula BCF, BCf, eſſe æqualia; nam anguli ad C ſunt recti, & latera BC, CF, æquantur lateribus BC, Cf, unde

* 4. El. 1. * $BF = Bf$.

Hinc BF, vel $Bf = CA$ vel Ca,

Nam $BF + Bf = CA + Ca$.

Fig. 13. Eodem modo oſtenditur eſſe,

Fb vel $bf = CA$ vel Ca. Unde conſtat,

1*mo*. Axem ſecundum Bb biſecari in C;

* 47. El. 1. Nam $BF = bF$. Unde * $\overline{BC}^2 + \overline{CF}^2 = \overline{bC}^2 + \overline{CF}^2$,

Ergo $\overline{BC}^2 = \overline{Cb}^2$, & $BC = Cb$.

2*do*. Axem ſecundum Bb primo Aa ſemper eſſe mi-

* 47. El. 1. norem; nam BC * minor eſt quàm BF, hoc eſt, quàm CA.

3*tio*. Si a termino B axis ſecundi Bb, tanquam centro, deſcribatur circulus radio BF ipſi CA æquali, iſte circulus axem primum A*a* in ipſis Ellipſeos focis F, *f*, ſecabit; rectæ enim, a puncto B ad focos F, *f*, ductæ, ipſi CA ſunt æquales: ſed radius circuli æquatur & ipſi CA: unde rectæ, a puncto B ad focos F, *f*, ductæ, ad circuli circumferentiam uſque pertingent: circulus igitur deſcriptus per focos tranſit.

COROLLARIUM IV.

41. HINC ex datis axibus A*a*, B*b*, facilè deſcribi poteſt Ellipſis; * inventis enim focis F, *f*, annectatur filum ejuſdem longitudinis cum Axe A*a*, & deſcribatur Ellipſis, ut in primâ definitione. * Art. 40.

COROLLARIUM V.

42. QUADRATUM dimidii axis ſecundi B*b*, æquatur rectangulo ſub AF & F*a*, partibus axis primi, inter focum F, & terminos A, *a*, comprehenſis.

Dico $\overline{BC}^2 = AF \times Fa$.

Sit CA vel * $BF = t$. $CF = m$. * Art. 40.

Et erit * $\overline{BC}^2 = \overline{BF}^2 - \overline{CF}^2 = tt - mm$, * 47. El. 1.

Sed $AF = AC - CF = t - m$,

Et $Fa = AC + CF = t + m$,

Unde $AF \times Fa = tt - mm = \overline{BC}^2$.

LEMMA II.

43. SICUT *Quatuor proportionales in æquationem facillimè convertuntur, ex eò quòd rectangulum ſub extremis* * *æ-* * 16. El. 6.

quetur rectangulo ſub mediis; ita viciſſim æquatio ad proportionem reducitur, habitis magnitudinibus ex unâ æquationis parte pro extremis, ex alterâ autem pro mediis.

Nam facta ex utrâque æquationis parte pro æqualibus rectangulis haberi poſſunt; & hæc rectangula latera
* 14. El. 6. habent * reciproca: iſtæ autem quantitates, quæ inter ſe ductæ factum ex alterutrâ æquationis parte conficiunt, pro rectanguli lateribus ſunt habendæ; unde termini in proportione inveniendâ innoteſcent.

PROPOSITIO I.

Theorema.

Fig. 12 & 13. 44. Si *ad axem primum* Aa *ducatur Ordinata* MP, *& ab eodem axe auferatur pars* AD, *ipſi* MF *æqualis; erit* CA : CF :: CP : CD.

Sit CA $= t$. CF $= m$. CP $= x$. PM $= y$. CD $= z$.

Fig. 12. Cadat Primò punctum P ſupra centrum C;

Erit igitur PF ſemper minor, quàm P*f*;

* 47. El. 1. Sed * $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$,

Et $\overline{Mf}^2 = \overline{MP}^2 + \overline{Pf}^2$,

Unde MF vel AD minor, quàm M*f* vel *a*D;

Ac proinde AD ſeu MF $=$ AC $-$ CD $= t - z$.

Et *a*D vel M*f* $=$ *a*C $+$ CD $= t + z$.

PF $=$ CF $-$ CP, vel CP $-$ CF $= m - x$ vel $x - m$, prout punctum P cadat infra vel ſupra focum F.

Erit etiam P*f* $=$ CP $+$ C*f* $= x + m$.

Jam verò, $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$.

Hoc eſt, $tt - 2tz + zz = yy + mm - 2mx + xx$;

Sed & $\overline{Mf}^2 = \overline{MP}^2 + \overline{Pf}^2$,

Hoc eſt, $tt + 2tz + zz = yy + mm + 2mx + xx$; ac proinde, ſi membra prioris Æquationis reſpectivè ſubducantur ex membris poſterioris, reſidua erunt æqualia;

Scilicet $4tz = 4mx$,

Et $tz = mx$,

Unde * $t : m :: x : z$, vel $\frac{mx}{t}$, * Art. 43.

Hoc eſt, CA : CF :: CP : CD $= z$ vel $\frac{mx}{t}$.

2*do*. CADAT Punctum P infra C, & erit ſemper PF major quam Pf; unde MF vel AD major, quàm Mf vel *a*D, ideoque erit Fig. 13.

AD ſive MF $=$ AC $+$ CD $= t + z$;

*a*D ſive Mf $=$ *a*C $-$ CD $= t - z$;

PF $=$ PC $+$ CF $= x + m$;

Pf $=$ PC $-$ Cf, vel Cf $-$ CP $= x - m$, vel $m - x$, prout punctum P cadat infra vel ſupra focum f;

Eſt autem $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$,

Hoc eſt, $tt + 2tz + zz = yy + mm + 2mx + xx$,

Et $\overline{Mf}^2 = \overline{MP}^2 + \overline{Pf}^2$.

Scilicet, $tt - 2tz + zz = yy + mm - 2mx + xx$, ac proinde ſi membra poſterioris æquationis reſpectivè ſubducantur ex membris prioris, reſidua erunt æqualia;

Scilicet $4tz = 4mx$,

Et $tz = mx$;

Ergo * $t : m :: x : z$, vel $\frac{mx}{t}$, * Art. 43.

Hoc eſt, CA : CF :: CP : CD $= z$ vel $\frac{mx}{t}$.

COROLLARIUM.

Fig. 12. 45. HINC, ſi fiat CA vel $Ca = t$, CF vel $Cf = m$; ſemper erit $MF = AC - CD = t - \frac{mx}{t}$, ita & $Mf = aC + CD = t + \frac{mx}{t}$, in primo caſu, quando P ſcilicet ſupra centrum C cadit.

Fig. 13. Cadente autem P infra C, erit ſemper $MF = AC + CD = t + \frac{mx}{t}$. Et $Mf = t - \frac{mx}{t}$.

PROPOSITIO II.

THEOREMA.

Fig. 12, & 13. 46. QUADRATUM *Ordinatæ Cujuſvis* MP *ad axem* Aa, *eſt ad rectangulum ſub abſciſſis* AP, P*a*, *ut ejus conjugati* B*b* *quadratum, ad ipſius* A*a* *quadratum*.

Dico $\overline{MP}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.

Sic $CA = t$. $CF = m$. $CP = x$. $PM = y$. $CD = z$.

Et erit $MF = AC - CD$, vel $AC + CD = t - z$, vel $t + z$, prout punctum P ſupra vel infra centrum C cadat.

Unde erit $\overline{MF}^2 = tt \mp 2tz + zz$.

Porro, $PF = CF - CP$, vel $CP - CF$, vel $CP + CF = x \mp m$,

Ideoque $\overline{PF}^2 = mm \mp 2mx + xx$,

Sed $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$,

Hoc eſt, $tt \mp 2tz + zz = yy + mm \mp 2mx + xx$.

Subſtituatur $\frac{mx}{t} = {}^{*}z$ in æquatione, loco ipſius z; & erit, * Art. 44.

$$tt \mp 2mx + \frac{mmxx}{tt} = yy + mm \mp 2mx + xx,$$

Ordinatis terminis prodit,

$$ttyy = t^4 - ttxx - mmtt + mmxx.$$

Unde $^{*}yy : tt - xx :: tt - mm : tt$, * Art. 43.

Hoc eſt, $\overline{MP}^2 : AP \times Pa :: {}^{*}\overline{BC}^2 : \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$. * Art. 42.

COROLLARIUM I.

47. QUADRATUM Ordinatæ cujuſcunque MK ad axem B*b*, eſt ad rectangulum ſub axis ipſius ſegmentis BK, K*b*, ut quadratum axis conjugati A*a*, ad quadratum ipſius B*b*.

Sit $Bb = 2c$. MK vel $CP = x$. CK vel $PM = y$. Et erit,

$$^{*}\overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2,$$ * Art. 46.

Hoc eſt, $yy : tt - xx :: 4cc : 4tt$,

Unde $^{*}\ 4ttyy = 4cctt - 4ccxx$, * 16. El. 6.

Vel $4ccxx = 4cctt - 4ttyy$,

Ergo $^{*}xx : cc - yy :: 4tt : 4cc$, * Art. 43.

Scilicet, $\overline{MK}^2 : BK \times Kb :: \overline{Aa}^2 : \overline{Bb}^2$.

COROLLARIUM II.

48. SIT Alteruter axium $Aa = 2t$. Ejus conjugatus $Bb = 2c$. Parameter autem ejus $= p$. Ordinatæ $PM = y$. Partes CP, inter centrum & Ordinatas, $= x$. His poſitis, *erit ſemper, Fig. 15. & 16. * Art. 46.

PM

$\overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2.$

Vel $yy : tt - xx :: 4cc : 4tt.$

* Def. 8. Sed * $Aa : Bb :: Bb : p,$

* Cor. 2. 20. Elem. 6. Unde * $\overline{Aa}^2 : \overline{Bb}^2 :: Aa : p;$

Ergo $\overline{Bb}^2 : \overline{Aa}^2 :: p : Aa.$

Ideoque $yy : tt - xx :: p : 2t.$

Ductis autem in ſe invicem primæ & poſtremæ proportionis terminis extremis & mediis, erit $yy = cc - \frac{ccxx}{tt}$, Ita & $yy = \frac{1}{2}pt - \frac{pxx}{2t}$.

Et cum hæc proprietas de omnibus Ellipſeos punctis vera ſit, atque eorum ſitum quoad axes conſtanter definiat, ſequitur æquationem $yy = cc - \frac{ccxx}{tt}$, ut & $yy = \frac{1}{2}pt - \frac{pxx}{2t}$, Ellipſeos naturam quoad axes perfectè & ad amuſſim exprimere & determinare.

COROLLARIUM III.

49. HINC quadratum Ordinatæ cujuſvis MP ad axem A*a*, eſt ad rectangulum ſub axis partibus AP, P*a*, ut parameter *p*, ad ipſum axem A*a*.

* Art. 48. Nam * $yy : tt - xx :: p : 2t,$

Hoc eſt $\overline{MP}^2 : AP \times Pa :: p : Aa.$

COR-

COROLLARIUM IV.

50. Si ad axem Aa ducantur duæ Ordinatæ MP, NQ; quadrata earum erunt ad ſe invicem, ut rectangula AP × Pa, & AQ × Qa, ſub axis partibus contenta.

Scilicet $\overline{MP}^2 : \overline{NQ}^2 :: AP \times Pa : AQ \times Qa$;

Nam *$\overline{MP}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$, * Art. 46.

Et $\overline{NQ}^2 : AQ \times Qa :: \overline{Bb}^2 : \overline{Aa}^2$,

Ergo $\overline{MP}^2 : AP \times Pa :: \overline{NQ}^2 : AQ \times Qa$,

Ideoque $\overline{MP}^2 : \overline{NQ}^2 :: AP \times Pa : AQ \times Qa$.

COROLLARIUM V.

51. Si per punctum quodvis P in alterutro axium conjugatorum Aa, agatur recta MM, alteri axi Bb parallela, ea Ellipſi in duobus punctis occurret a puncto P æquè remotis,

RECTAM MM Ellipſi occurrere manifeſtum eſt, ex eò quod axis Bb ei occurrat in punctis, B, b; ideoque *erunt MP, MP, ordinatæ; ac proinde earundem quadrata $\overline{MP}^2$, $\overline{MP}^2$, eidem quantitati $cc - \frac{ccxx}{tt}$ æquantur, ergo MP = MP. * Def. 7.

COROLLARIUM VI.

52. Cum ſit *$yy = cc - \frac{ccxx}{tt}$, liquet, quo major fuerit CP, vel x, ex utravis parte centri C, hoc eſt, quò major * Art. 48.

jor fuerit quantitas ſubducenda $\frac{ccxx}{tt}$, eo minorem fore PM vel y; adeo ut, ſi CP vel x ipſi CA vel t fuerit æqualis, hoc eſt, ſi quantitas $\frac{ccxx}{tt}$ æqualis ſit ipſi cc, prorſus evaneſcat PM vel y; & è contrario, quò minor fuerit CP vel x, eò major erit PM vel y; ideoque evaneſcente CP vel x, Ordinata PM (quæ in hoc caſu fit CB) maxima erit omnium ad axem A*a* Ordinatarum. Unde patet,

1*mo*. SI per terminos B, *b*, axis unius B*b*, ducantur rectæ alteri A*a* parallelæ, has eſſe tangentes in iſtis punctis B, *b*; & contra.

2*do*. PUNCTA Ellipſeos inter A & B comprehenſa, eo longius ab axe AP diſtare, quo remotiora ſint a puncto A; illa autem, quæ inter B & *a* contineantur, axi eò eſſe propiora, quo remotiora ſint a puncto B.

COROLLARIUM VII.

53. SI puncta P & P æquè remota ſint a centro C, Ordinatæ MP, MP, ex eâdem axis parte ſumptæ, erunt æquales; cadente enim puncto P ſupra centrum C, erit $\overline{PM}^2$ vel *$yy = cc - \frac{ccxx}{tt}$; erit etiam, puncto P infra C cadente, $\overline{PM}^2$ vel $yy = cc - \frac{ccxx}{tt}$; Sed cum CP vel x ex altera centri parte æqualis ſit ipſi CP vel x ex altera, & cum cc & tt quantitates ſint conſtantes, erit $cc - \frac{ccxx}{tt} = cc - \frac{ccxx}{tt}$, unde $yy = yy$; vel PM = PM. Ideoque

* Art. 48.

ſi recta MM Ellipſi utrinque occurrens, ab uno conjugatorum axium B*b* biſariam ſecetur in puncto K, ea erit alteri conjugato A*a* parallela.

Ductis enim ad axem B*b* parallelis MP, MP, conſtat rectam PP biſariam *ſecari in puncto C, quia MM *2 Elem. 6. biſecatur in K; ideoque PM, PM, erunt Ordinatæ æquales; unde *recta MM ipſi A*a* parallela erit. *33. El. 1.

Corollarium VIII.

54. Si intelligamus Ellipſeos partem BA*b* parti B*ab* ſuperimponi, hæc illi perfectè congruet, hoc eſt, puncta A & M cadent ſuper puncta *a* & M, cum omnes perpendiculares A*a*, MM, biſariam ſecentur in punctis C & K; eodem modo probatur partem AB*a* ipſi A*ba* perfecte congruere & eſſe æqualem: unde liquet Ellipſin a duobus axibus ſecari in quatuor partes æquales, & ſimiles, ſitu tamen diverſo. Nam *b*AB & AB*a* ſingulæ ſemiellipſin conficiunt; auferatur communis pars ABC, & erit A*b*C = *a*BC. Simili ratiocinio conſtabit, partem ABC æqualem eſſe parti *ab*C.

PROPOSITIO III.

Theorema.

55. Si *per terminum* A *ipſius Axis* A*a*, *agatur recta* LAL *Conjugato* B*b* *parallela, & in utrovis angulorum* *a*AL, *a*AL, *recta quæcunque* AM; *dico hanc rectam* AM *Ellipſi iterum occurrere in alio puncto* M.

Sumpta enim ſuper AL, ex alterutra parte puncti A; Fig. 17. recta AG ipſius axis A*a* Parametro *p* æquali; agatur GF ipſi A*a* parallela, rectæ AM (productæ, ſi opus) in puncto F occurrens. Capiatur etiam ſuper AL (ex eâdem axis parte, quà AM) pars AL ipſi GF æqualis, & ab altero axis Aa termino a ducatur aL; dico punctum M, ubi hæc recta ipſi AM occurrat, eſſe in Ellipſi MAM.

Sit enim MP ipſi AL parallela, & A$a = 2t$. AG $= p$. GF vel AL $= a$. CP $= x$. PM $= y$. Triangula

* Art. 15. autem AGF, MPA, ſunt * ſimilia.

Unde AG : GF :: MP : PA,

Hoc eſt $p : a :: y : \frac{ay}{p}$ Ergo AP $= \frac{ay}{p}$.

* 2. El. 6. Porro ob triangula LAa, MPa * ſimilia erit,

AL : Aa :: PM : Pa.

Scilicet $a : 2t :: y : \frac{2ty}{a}$, Unde P$a = \frac{2ty}{a}$,

Erit igitur AP × P$a = \frac{2tyy}{p}$.

Sed AP $= t \mp x$. Et P$a = t \pm x$, Prout punctum P ſupra vel infra centrum C cadit.

Ergo & AP × P$a = tt - xx$,

Ideoque $\frac{2tyy}{p} = tt - xx$,

Et $yy = \frac{1}{2}pt - \frac{pxx}{2t}$;

* Art. 48. * Erit igitur PM Ordinata ad axem Aa, hoc eſt, punctum M eſt ad Ellipſin.

Corollarium I.

56. Hinc ex dato Ellipſeos axe Aa, parametro p, ductâque per iſtius axis verticem A recta AM, in utrovis angulorum aAL, aAL, facilè innoteſcit punctum M.

Cor-

COROLLARIUM II.

57. MANIFESTUM eſt, nullam aliam rectam, præter LAL, Ellipſin tangere poſſe in puncto A; ea enim unica eſt, quæ per punctum A ducta, ipſam LL non ſecet; omnes igitur aliæ in angulo PAL neceſſariò cadent, ideoque Ellipſi occurrent in alio puncto M.

PROPOSITIO IV.

THEOREMA.

58. OMNES *Diametri* MC*m* *biſariam ſecantur in Centro* C, *& Ellipſi in duobus tantum punctis* M & *m* *occurrunt.* Fig. 17.

DUCTA Ordinatâ MP, ſumatur C*p* ipſi CP æqualis; ex puncto autem *p* ducta *pm* ipſi C*p* normalis, diametro MC*m* occurrat in *m*; triangula igitur CPM, C*pm* ſunt *ſimilia, & † æqualia; unde CM = C*m*, & PM = *pm*. * 4. El. 6. † 26. El. 1. Sed Ordinatæ a centro C æquidiſtantes ſunt *æquales, * Art. 53. & PM eſt ex Hypotheſi Ordinata, ideoque *pm* erit etiam Ordinata, & punctum *m* eſt in Ellipſi.

PORRO ſi fingamus rectam aliquam ipſi B*b* parallelam a puncto C verſus A moveri, liquet partem parallelæ intra angulum ACM contentam conſtanter augeri, partem verò inter rectam CM, & Ellipſeos portionem AM*b* incluſam, Ordinatam ſcilicet PM, continuo *diminui; * Art. 52. unde patet rectam CM ad Ellipſin magis magiſque accedere, uſque dum ei occurrat in puncto M, poſtea autem magis magiſque recedere; non igitur recta CM Ellipſi iterum occurrit in alio puncto præter M ex eâdem axis parte; & cum idem oſtendi poſſit de recta

Cm, ſequitur Diametrum MCm Ellipſi in duobus tantum punctis M & m occurrere.

DEFINITIONES.

XII.

Fig. 18, 19, 20, 21. SI per Ellipſeos punctum quodvis M ducantur Diameter MCm, Ordinata MP ad alterutrum axem Aa, & recta quædam MT ita, ut ſit CT tertia proportionalis ad CP, CA; Diameter SCs ipſi MT parallela, dicitur eſſe Diametro MCm CONJUGATA; & viciſſim, Diameter Mm CONJUGATA eſt ipſi Ss, adeo ut Mm, Ss, ſimul ſumptæ appellentur DIAMETRI CONJUGATÆ.

XIII.

OMNES rectæ ab Ellipſeos punctis ductæ, uni ex diametris conjugatis parallelæ, & ab altera terminatæ, vocantur ORDINATÆ ad illam alteram; ita NO, Diametro Ss parallela, eſt ORDINATA ad Conjugatam ejus Mm.

XIV.

TERTIA proportionalis ad duas Diametros Conjugatas, vocatur PARAMETER primi in proportione termini; tertia ſcilicet proportionalis ad Mm, Ss, PARAMETER eſt Diametri Mm.

COROLLARIUM.

* Def. 12. 59. SIT CA $= t$, CP $= x$; PT $= s$. Et * erit,

$$CP : CA :: CA : CT,$$

Hoc eſt, $x : t :: t : x + s$;

* 16. El. 6. Sed * $x : t :: t : \frac{tt}{x}$.

Ergo

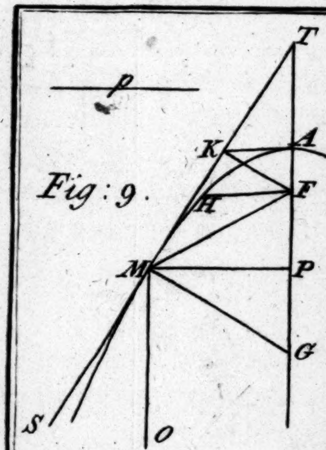
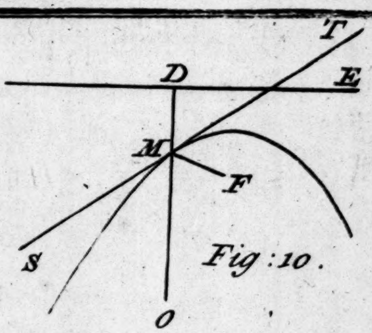


Fig: 9.



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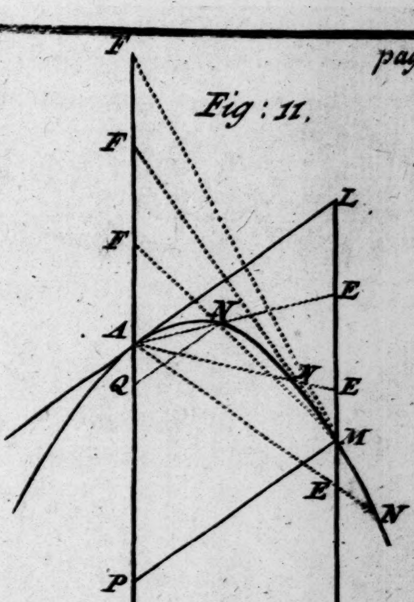

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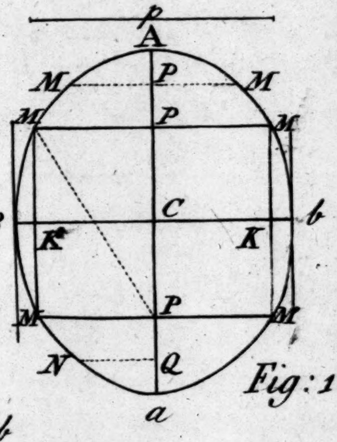
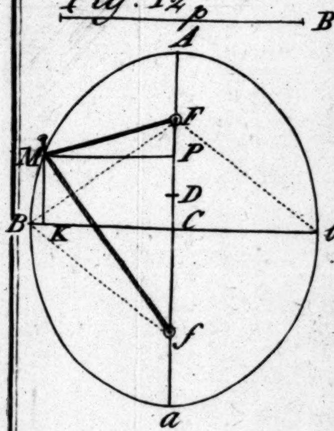


Fig: 16

Fig: 12



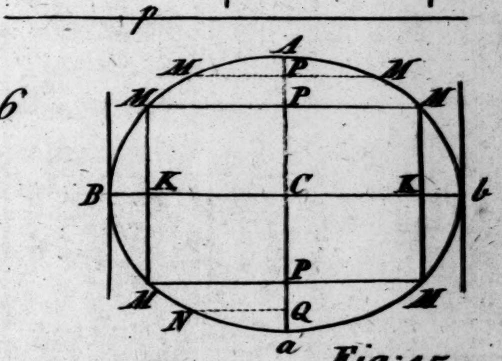


Fig: 15

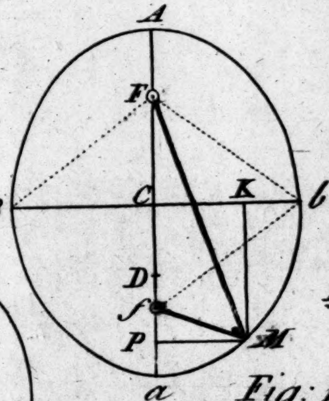


Fig: 13

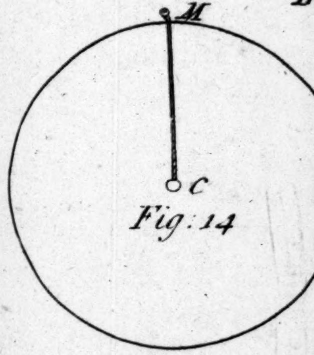


Fig: 14

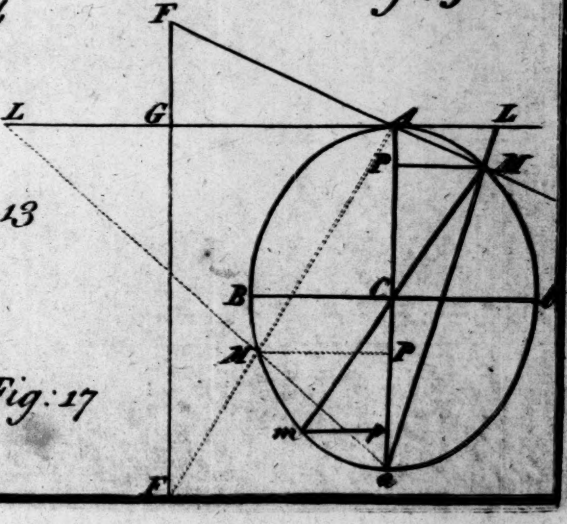
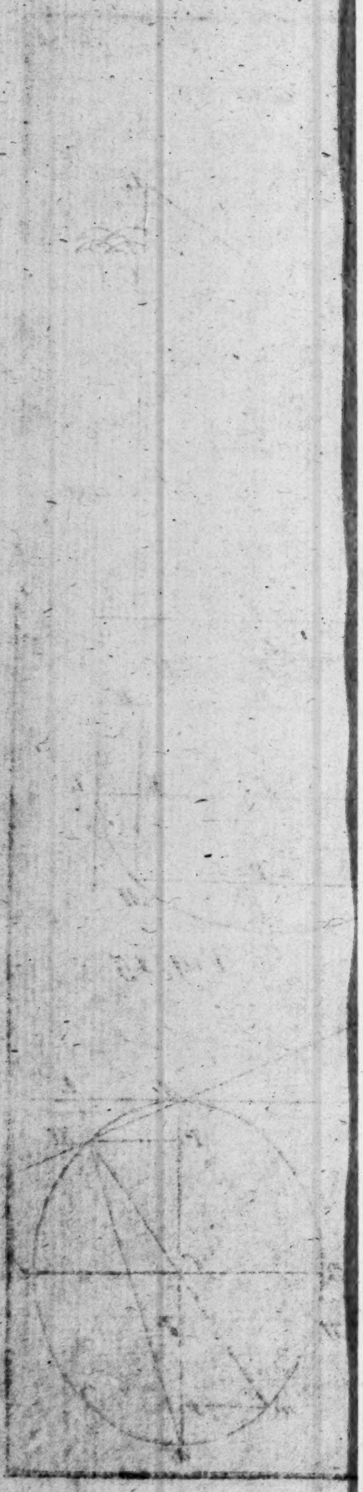


Fig: 17



Ergo $x + s = \frac{tt}{x}$.

Vel $sx = tt - xx = AP \times Pa$.

PROPOSITIO V.

THEOREMA.

60. SI *per terminos* M, S, *duarum Diametrorum Conjugatarum* Mm, Ss, *agantur ad Axem* A*a* *Ordinatæ* MP, SK; *dico Abſciſſam* CK *eſſe mediam proportionalem inter ſegmenta axis ab alterius Ordinatæ* MP *concurſu facta.*

Scilicet * $\overline{CK}^2 = AP \times Pa$, * 17. El. 6.

Sit CA $= t$; CP $= x$; PT $= s$. CK $= m$. Et erit,

$AP \times Pa = tt - xx =$ * sx. * Art. 59.

Et $AK \times Ka = tt - mm$,

Sed * $tt = sx + xx$; * Art. 59.

Unde $AK \times Ka = sx + xx - mm$;

Hoc poſito * erit, * Art. 50.

$AP \times Pa : AK \times Ka :: \overline{PM}^2 : \overline{KS}^2$;

Sed $\overline{PM}^2 : \overline{KS}^2 :: \overline{TP}^2 : \overline{CK}^2$ ob triangula TPM, CKS, * ſimilia; * Art. 15.

Unde $AP \times Pa : AK \times Ka :: \overline{TP}^2 : \overline{CK}^2$,

Hoc eſt, $sx : sx + xx - mm :: ss : mm$.

Ergo * $mmsx = sssx + ssxx - mmss$, dividendo per s, & tranſponendo erit, * 16. El. 6.

$$mmx + mms = sxx + ssx;$$

Vel $mm = sx$,

Hoc eſt $\overline{CK}^2 = AP \times Pa$.

COR-

COROLLARIUM.

61. SUMMA quadratorum ex duabus diametris conjugatis M*m*, S*s*, æquatur ſummæ quadratorum ex duobus axibus A*a*, B*b*.

Dico $\overline{Mm}^2 + \overline{Ss}^2 = \overline{Aa}^2 + \overline{Bb}^2$.

* 5. El. 2. Sit $CA = t$; $Bb = 2c$; $CP = x$. Et *erit

$\overline{CA}^2 - \overline{CK}^2 = AK \times Ka$.

* Art. 60. Eſt autem $\overline{CA}^2 = tt$; & *$\overline{CK}^2 = AP \times Pa = tt - xx$.

Unde $\overline{CA}^2 - \overline{CK}^2$, vel $AK \times Ka = xx$.

* Art. 48. Porro * $\overline{CA}^2 : \overline{CB}^2 :: AK \times Ka : \overline{SK}^2$.

Scilicet $tt : cc :: xx : \frac{ccxx}{tt} = \overline{SK}^2$;

Et $\overline{CA}^2 : \overline{CB}^2 :: AP \times Pa : \overline{PM}^2$,

Vel $tt : cc :: tt - xx : cc - \frac{ccxx}{tt} = \overline{PM}^2$,

* 47. El. 1. Jam vero * $\overline{CM}^2 = \overline{CP}^2 + \overline{PM}^2 = xx + cc - \frac{ccxx}{tt}$,

Et $\overline{CS}^2 = \overline{CK}^2 + \overline{KS}^2 = tt - xx + \frac{ccxx}{tt}$,

Unde $\overline{CM}^2 + \overline{CS}^2 = tt + cc$ (quantitatibus + & — ſe mutuo deſtruentibus,)

Sed $\overline{CA}^2 + \overline{CB}^2 = tt + cc$,

Ergo $\overline{CM}^2 + \overline{CS}^2 = \overline{CA}^2 + \overline{CB}^2$,

Hoc eſt $\overline{Mm}^2 + \overline{Ss}^2 = \overline{Aa}^2 + \overline{Bb}^2$,

PRO-

PROPOSITIO VI.

THEOREMA.

62. QUADRATUM *Ordinatæ cujuſvis* ON *ad Diametrum* Mm, *eſt ad rectangulum* MO × Om *ſub abſciſſis, ut quadratum ejus Conjugatæ* Ss, *ad quadratum ipſius Diametri* Mm.

$$\text{Dico } \overline{ON}^2 : MO \times Om :: \overline{Ss}^2 : \overline{Mm}^2.$$

Ductis enim NQ, OH ad axem B*b*, & OR ad ejus conjugatum A*a* parallelis, occurrat OR Ordinatæ NQ (productæ, ſi opus) in puncto R.

Sit $CP = x$; $PM = y$; $CA = t$; $PT = s$; HQ vel $OR = a$. $CH = b$. Triangula autem CPM, CHO ſunt *ſimilia, *2. El. 6.

Unde CP : PM :: CH : HO,

$$\text{Vel } x : y :: b : \frac{by}{x} = HO \text{ vel } RQ.$$

Triangula etiam MPT, NRO ſunt *ſimilia, *Art. 15.

Unde TP : PM :: OR : RN.

$$\text{Vel } s : y :: a : \frac{ay}{s} = RN.$$

His poſitis erit in figuris 18, 19,

$$NQ = RQ - RN, \text{ vel } RN - RQ = \frac{by}{x} - \frac{ay}{s} \text{ vel } \frac{ay}{s} - \frac{by}{x},$$

$$\text{Unde } \overline{NQ}^2 = \frac{bbyy}{xx} - \frac{2abyy}{sx} + \frac{aayy}{ss}.$$

$$\text{Et in fig. 20, 21. } NQ = RQ + RN = \frac{by}{x} + \frac{ay}{s},$$

$$\text{Ergo } \overline{NQ}^2 = \frac{bbyy}{xx} + \frac{2abyy}{sx} + \frac{aayy}{ss}.$$

Ideoque in figuris 18, 19, 20, & 21, erit,

$$\overline{NQ}^2 = \frac{bbyy}{xx} \mp \frac{2abyy}{sx} + \frac{aayy}{ss}.$$

Rurſus in Fig. 18, 19.

$$CQ = CH + HQ = b + a,$$

Et in Fig. 20, 21,

$$CQ = HQ - CH \text{ vel } CH - HQ = a - b \text{ vel } b - a,$$

ac proinde in figuris 18, 19, 20, 21,

Erit $\overline{CQ}^2 = aa \pm 2ab + bb.$

* Art. 50. Porro * $AP \times Pa : AQ \times Qa :: \overline{PM}^2 : \overline{QN}^2,$

Vel $tt - xx : tt - aa \mp 2ab - bb :: yy : \frac{ttyy - aayy \mp 2abyy - bbyy}{tt - xx};$

* 5. El. 2. (nam *$AQ \times Qa = \overline{CA}^2 - \overline{CQ}^2 = tt - aa - 2ab - bb.$)

Unde $\overline{NQ}^2 = \frac{ttyy - aayy \mp 2abyy - bbyy}{tt - xx};$

Sed $\overline{NQ}^2 = \frac{bbyy}{xx} \mp \frac{2abyy}{sx} - \frac{aayy}{ss},$

Ergo $\frac{bbyy}{xx} \mp \frac{2abyy}{sx} + \frac{aayy}{ss} = \frac{ttyy - aayy \mp 2abyy - bbyy}{tt - xx};$

* Art. 50. Eſt autem $\mp \frac{2abyy}{xx} = \mp \frac{2abyy}{tt - xx}$, nam *$sx = tt - xx.$

Unde $\frac{bbyy}{xx} + \frac{aayy}{ss} = \frac{ttyy - aayy - bbyy}{tt - xx}$; dividendo per yy,

Erit $\frac{bb}{xx} + \frac{aa}{ss} = \frac{tt - aa - lb}{tt - xx}.$

Ductâ autem hâc poſtremâ æquatione in xx, & tranſlatâ bb ad alteram partem, erit,

$$\frac{aaxx}{ss} \text{ vel } \frac{aax^4}{ssxx} = \frac{ttxx - aaxx - bbxx}{tt - xx} - bb.$$

Vel $\frac{aax^4}{ssxx} = \frac{ttxx - aaxx - bbtt}{tt - xx}$ (reductâ ſcilicet bb ad eandem denominationem cum fractione) unde (ſi ducatur primum æquationis membrum *viz.* $\frac{aax^4}{ssxx}$ in $ssxx$; alterum vero, in quadratum ipſius $tt - xx$, ipſi sx * æqualis; vel, quod eodem recidit, ſi numerator fractionis $\frac{ttxx - aaxx - bbtt}{tt - xx}$ in ſimplicem $tt - xx$ multiplicetur) erit, * Art. 59.

$$aax^4 = t^4xx - aattxx - bbt^4 - ttx^4 + aax^4 + bbttxx;$$

& auferendo utrinque aax^4, tranſponendo $-aattxx$, & dividendo per $ttxx$, evadit,

$$aa = tt - xx + bb - \frac{bbtt}{xx},$$

Hoc eſt, $\overline{OR}^2$ vel $\overline{HQ}^2 = tt - xx + bb - \frac{bbtt}{xx}$.

Jam vero ſi fiat CM vel $Cm = z$; erit, ob triangula CPM, CHO *ſimilia, * 2. El. 6.

CP : CM :: CH : CO,

Vel $x : z :: b : \frac{bz}{x} = CO$.

Unde $MO \times Om = * \overline{CM}^2 - \overline{CO}^2 = \frac{xxzz - bbzz}{xx}$. * 5. El. 6.

Porro, triangula ORN, CKS, ſunt *ſimilia, * Art. 15.

Unde $\overline{ON}^2 : \overline{CS}^2 :: \overline{OR}^2 : \overline{CK}^2$,

Hoc eſt, $\overline{ON}^2 : \overline{CS}^2 :: tt - xx + bb - \frac{bbtt}{xx} : * tt - xx$; * Art. 60.

Sed $* tt - xx + bb - \frac{bbtt}{xx} : tt - xx :: \frac{xxzz - bbzz}{xx} : zz$, * 16. El. 6.

Unde $\overline{ON}^2 : \overline{CS}^2 :: MO \times Om : \overline{CM}^2$,

Permutando $\overline{ON}^2 : MO \times Om :: \overline{CS}^2 : \overline{CM}^2 :: \overline{Ss}^2 : \overline{Mm}^2$.

COROLLARIUM UNIVERSALE.

63 MANIFESTUM eſt quæcunque fuerint demonſtrata in Propoſitione ſecundâ, de duobus axibus Aa, Bb; ope hujus Propoſitionis locum habere apud duas quaſvis Diametros Conjugatas Mm, Ss; & cum articuli 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, a ſecundâ Propoſitione clarè deducantur, verique ſint, utrum Angulus ACB rectus ſit necne, conſtat, ſi ponamus rectas Aa, Bb, in iſtis Articulis non eſſe axes, ſed duas diametros conjugatas quaſcunque, eos adhuc veros fore, demonſtratio enim eorum prorſus eadem erit.

COROLLARIUM II.

Fig. 22. 64. CUM articuli 52 & 57, æque veri ſint, ſive Aa, Bb axes fuerint, ſive Diametri conjugatæ quæcunque Mm, Ss; ſequitur, lineam MT per terminum M Diametri cujuſvis Mm ductam, ipſi autem Ss conjugatæ cum Mm parallelam, eſſe tangentem in puncto M, eamque eſſe unicam, quæ Ellipſin in illo puncto tangere poſſit. Et vice versâ.

Unde conſtat, unam ſolummodo rectam Ellipſin in dato puncto tangere poſſe.

COROLLARIUM III.

65. HINC, ſi per Ellipſeos punctum quodvis M, agatur ad alterutrum axem Aa Ordinata quævis MP, & ſumatur recta CT, ex parte puncti P, tertia Proportionalis ad CP, CA, & jungatur MT, liquet hanc rectam MT eſſe tangentem in puncto M; nam MT diametro Ss eſt * parallela; ac proinde Ellipſin in puncto M * tangit.

* Def. 12.
* Art. 64.

Et vice verſâ, ſi recta MT Ellipſin in puncto M tangat, & ducatur ad alterutrum axem A*a* Ordinata MP; erunt partes CP, CA, CT continuè proportionales; ideoque CP × CT = $\overline{CA}^2$.

COROLLARIUM IV.

66. Si in definitionibus 12, 13 & 14, & in duabus ultimis Propoſitionibus ponamus rectas A*a*, B*b*, non eſſe Axes, ſed duas diametros quaſvis conjugatas, conſtabit adhuc harum Propoſitionem veritas, eadem enim erit demonſtratio, ut ex ſolâ Figuræ 22 inſpectione patet, cum ex ſimilibus triangulis CPM, CHO, & MPT, NRO, eadem eliciatur proportio, ac habitis A*a*, B*b*, pro Axibus Ellipſeos.

Unde liquet conſectarium præcedens æqualiter obtinere, ſive A*a* fuerit Axis, ſive alia quævis Diameter;

Patet etiam, Diametros Conjugatas M*m*, S*s*, ex hac Hypotheſi, pro axibus haberi poſſe; ideoque duos axes pro diametris habendos, quæ inter ſe rectos conſtituant angulos.

PROPOSITIO VII.

THEOREMA.

67. Si, *in Ellipſi ducantur diametri quævis conjugatæ* Mm, Ss, *axes autem* Aa, Bb; *dico parallelogrammum ſub diametris* Mm, Ss, *æquari rectangulo ſub axibus* Aa, Bb. Fig. 23.

Per puncta M, S, agantur rectæ MD, SD ipſis Ss, Mm parallelæ, ſibi mutuo occurrentes in D; producta autem MD occurrat axi A*a* producto in T; & a puncto M demittatur MP ipſi A*a* normalis, & *erunt MD, *Art. 52. 63.
SD tangentes in punctis M, S, & CMSD parallelogrammum ſub ſemidiametris conjugatis CM, CS, & *æ- *36. El. 1.

quale quartæ parti parallelogrammi circumſcripti & ſub ipſis diametris Mm, Ss, contenti, ex eo quod Mm, Ss
* Art. 58. *bifariam ſecentur in centro C. Demittatur CE ipſi DM
* Art. 47. normalis, & erit rectangulum CS × CE * æquale parallelogrammo CD; dico igitur eſſe CS × CE $=$ CA × CB $= \frac{1}{4}$ rectanguli ſub axibus; ſit CA $= t$. CB $= c$. CP $= x$.

* Art. 46. Et erit *$\overline{CA}^2 : AP \times Pa :: \overline{CB}^2 : \overline{MP}^2$,

Hoc eſt $tt : tt - xx :: cc : cc - \frac{ccxx}{tt} = \overline{MP}^2$.

* 47. El. 1. Unde * $\overline{CM}^2 = xx + cc - \frac{ccxx}{tt}$.

* Art. 65. Porro * $CP : CA :: CA : CT$,

Vel $x : t :: t : \frac{tt}{x}$. Ergo $PT = CT - CP = \frac{tt}{x} - x$,

$$\text{Et } \overline{PT}^2 = xx - 2tt + \frac{t^4}{xx}.$$

$$\text{Et } \overline{MT}^2 = xx + cc - 2tt + \frac{t^4}{xx} = \frac{ccxx}{tt}.$$

* 4 El. 6. Triangula autem MPT, CET ſunt * ſimilia ob angulos ad E & P rectos, & ob angulum ad T communem; unde

$$\overline{MT}^2 : \overline{MP}^2 :: \overline{CT}^2 : \overline{CE}^2,$$

$$\text{Vel } xx + cc - 2tt + \frac{t^4}{xx} - \frac{ccxx}{tt} : cc - \frac{ccxx}{tt} :: \frac{t^4}{xx} : \frac{t^4cc}{t^4 - ttxx + ccxx}$$

$$\text{Unde } \overline{CE}^2 = \frac{t^4cc}{t^4 - ttxx + ccxx}.$$

* Art. 61. Porro * $\overline{CS}^2 = \overline{CA}^2 + \overline{CB}^2 - \overline{CM}^2 = tt - xx + \frac{ccxx}{tt}$.

$$\text{Et } \overline{CE}^2 \times \overline{DM}^2 = \frac{t^8cc + t^4c^4xx - t^6ccxx}{t^6 + t^2c^2x^2 - t^4x^2} = ttcc,$$

Sed $ttcc = \overline{CA}^2 \times \overline{CB}^2$,

Ergo $\overline{CE}^2 \times \overline{CS}^2 = \overline{CA}^2 \times \overline{CB}^2$, nam $\overline{DM}^2 = \overline{CS}^2$;

Vel CE × CS, hoc eſt parallelogrammum CD = CA × CB.

COROLLARIUM.

68. Hinc totum parallelogrammum circa diametros M*m*, S*s*, deſcriptum, æquatur toti rectangulo ſub axibus A*a*, B*b*; ideoque parallelogramma omnia, circa datæ Ellipſeos diametros quaſvis conjugatas deſcripta, erunt inter ſe æqualia; utpote quæ eidem rectangulo ſub axibus facto æquentur.

PROPOSITIO VIII.

THEOREMA.

69. In *Ellipſi, cujus centrum* C, *ſi per punctum quodvis* M *agatur ad axem* Aa *ordinata* MP, *& ad tangentem* MT *per idem punctum ductam normalis quædam* MG; *dico eſſe ſemper* CP *ad* PG *in ratione data axis ipſius* Aa *ad parametrum ejus* p. Fig. 24.

Scilicet CP : PG :: A*a* : *p*.

Sit CA vel C*a* $= t$; CP $= x$; PM $= y$; & erit

*CT $= \frac{tt}{x}$, unde PT $= \frac{tt-xx}{x}$. * Art. 59.

Jam triangula TPM, MPG, ſunt *ſimilia, * 8. El. 6.

Ergo TP : PM :: PM : PG,

Vel $\frac{tt-xx}{x} : y :: y : \frac{xyy}{tt-xx} =$ PG;

Unde * $x : \frac{xyy}{tt-xx} :: tt - xx : yy$, * 16. El. 6.

Hoc eſt, CP : PG :: AP × P*a* : $\overline{PM}^2$,

Sed

* Art. 49. Sed $*AP \times Pa : \overline{PM}^2 :: Aa : p$,
Ergo $CP : PG :: Aa : p$.

COROLLARIUM I.

70. Hinc rectangulum $MP \times PC : \overline{PM}^2 :: Aa : p$. nam, iiſdem poſitis ac in propoſitione, erit $TP \times PC = tp - xx$.

* Art. 49. Sed $*yy : tt - xx :: p : 2t$,
Vel $tt - xx : yy :: 2t : p$.

Hoc eſt $TP \times PC : \overline{PM}^2 :: Aa : p$.

PROPOSITIO IX.

THEOREMA.

Fig. 25, & 26. 71. Si *in Ellipſi ab extremitatibus* A, a, *Diametri cujuſvis* Aa, *agantur rectæ* AO, ag, *Ordinatis parallelæ, & alia quæpiam recta* MT *quomodocunque contingens ducatur, dico rectangulum ſub rectis* AG, ag, *æquari quadrato dimidiæ Diametri* CB, *ſemiconjugatæ cum* Aa.

Scilicet $AG \times ag = \overline{CB}^2$.

Fig. 25. Si tangens Gg per extremitatem B tranſit, res conſtat; nam ob parallelas, $AG \times ag = \overline{CB}^2$.

Fig. 26. Non autem tranſeat per B; ſed Cg, Aa, productæ conveniant in T; ducanturque per punctum M recta MP ipſi AG, MO ipſi Aa parallela;

* Art. 66. Quoniam *igitur $TC : CA :: CA : CP$,
* 19. El. 5. Erit $*TC : CA :: TA : AP$;
* Art. 58. Sed $*CA = Ca$,

Ergo $Ca : TC :: AP : AT$,
Et * $Ta : TC :: TP : AT$, *18. El. 5.
Unde * $ag : CN :: PM : AG$, *2 Elem. 6.
Et * $AG \times ag = PM \times CN$, *16. El. 6.
Sed $PM \times CN = CN \times CO = {}^{*}\overline{CB}^{2}$, *Art. 65. & 66.
Ergo $AG \times ag = \overline{CB}^{2}$.

PROPOSITIO X.

THEOREMA.

72. SI *per Ellipſeos punctum quodvis* M *agantur ad* Fig. 27. *focos* F, *f*, *rectæ* MF, M*f*; *& per idem punctum tangens* TMS; *dico angulum* FMT *ab unâ tangentis parte* TM *& rectâ* MF *factum, æquari angulo* *f*MS *ſub alterâ tangentis parte* MS *& rectâ* M*f* *comprehenſo.*

Scilicet angulum $FMT = fMS$,

Ductis enim ex punctis F, *f*, rectis FD, *fd*, tangenti TMS perpendicularibus, producatur axis A*a*, donec tangenti occurrat in T, & ſit MP Ordinata ad axem.

Sit CA vel $Ca = t$; CF vel $Cf = m$; $CP = x$;

Et * erit $t - \frac{mx}{t} : t + \frac{mx}{t} :: \frac{tt}{x} - m : \frac{tt}{x} + m$, *16. El. 6.

Hoc eſt, * $MF : Mf :: TF$ vel $CT - CF : Tf$. * Art. 45.
Sed * $TF : Tf :: FD : fd$, *2. El. 6.
Unde $MF : Mf :: FD : fd$; ac proinde triangula FMD, *f*M*d*, erunr * ſimilia; ideoque anguli FMD, *f*M*d*, vel *7. El. 6. FMT, *f*MS, lateribus homologis DF, *df* oppoſiti, erunt æquales.

COR-

COROLLARIUM I.

73. HINC, ſi ab Ellipſeos puncto quovis M ducantur ad focos F, f, rectæ MF, Mf; & per idem punctum M recta TMS, ita ut angulus FMT æqualis ſit angulo fMS; conſtat rectam TMS Ellipſin tangere in puncto M.

COROLLARIUM II.

Fig. 28. 74. SI ab Ellipſeos focis F, f, ad punctum quodvis tertium V inflectantur duæ rectæ FV, fV, quarum una fV axi majori Aa æqualis ſit, altera FV a perpendiculo TMS in ſe demiſſo biſecetur in T, perpendiculum illud TMS Ellipſin tanget:

Secet enim perpendiculum TMS rectam fV (productam) in puncto M, & jungatur FM; ergo ob æquales TV, TF, & communem TM, erit latus
* 4. El. 6. *VM = MF, & angulus TMV = TMF. Unde erit MF
* Hyp. + Mf = MV + Mf = *Aa, axi majori: ideoque punctum
* Art. 39. M * eſt in Ellipſi. Sed angulus TMF = TMV: &
* 15. El. 1. *fMS = TMV; unde TMF = fMS, & recta TMS
* Art. 73. Ellipſin * tangit in puncto M.

Et contra, ſi recta TMS Ellipſin tangit, erit fV æqualis Axi majori. Nam ob tangentem TMS erit an-
* Art. 72. gulus * FMT = fMS. Unde & FMT = TMV; ſed anguli ad T ſunt æquales, utpote recti. Unde triangula
* 26. El. 1. FMT, VMT * æquantur, ac proinde FM = MV. Sed
* Art. 39. FM + Mf = * Aa, axi majori; ergo MV + Mf = Vf = Aa.

PRO-

pag. 46

Fig: 18

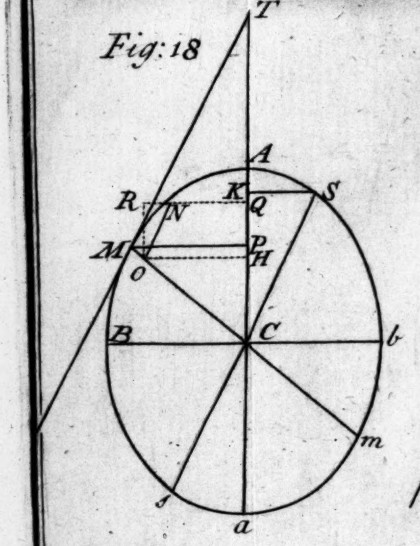


Fig: 19.

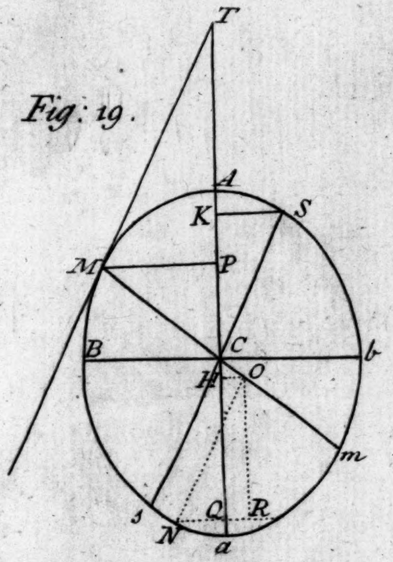


Fig: 20

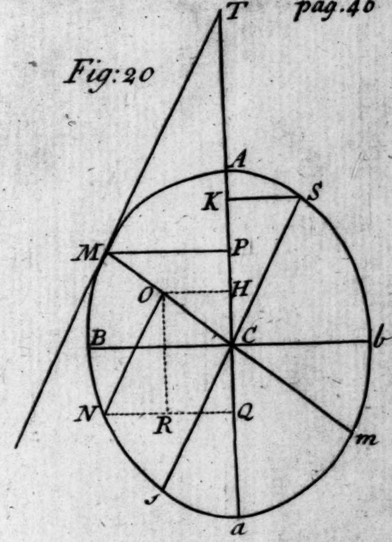


Fig: 21

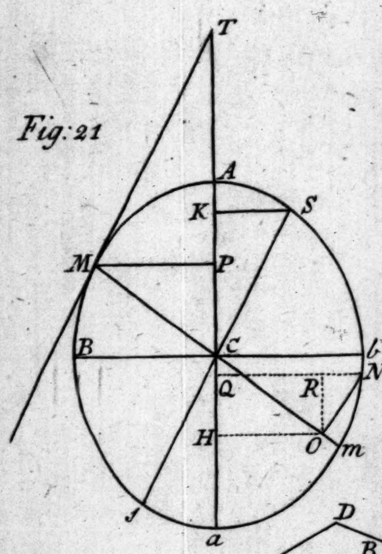


Fig: 22

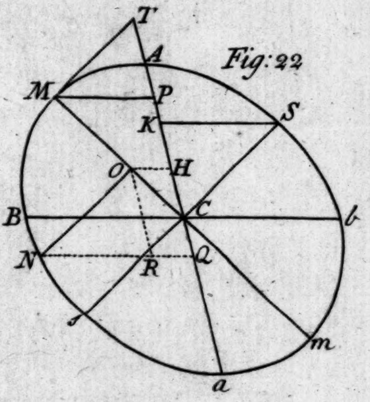


Fig: 24

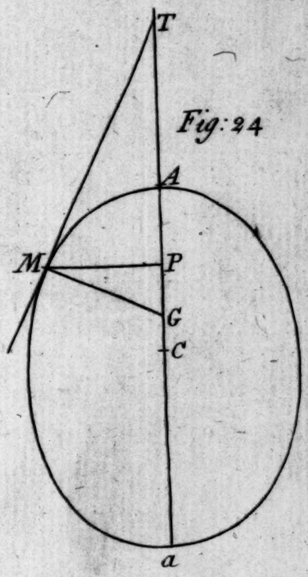
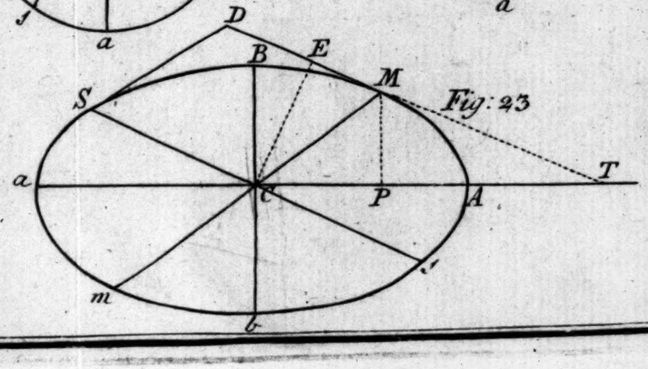


Fig: 23



PROPOSITIO XI.

THEOREMA.

75. SI *per alterutrum terminum* A *Diametri* A*a* *ducatur recta* DAE, *conjugato ejus parallela, duabus aliis conjugatis diametris* M*m*, S*s*, *in punctis* D *&* E *occurrens; dico rectangulum ſub iſtius parallelæ ſegmentis*, DA, AE, *æquari quadrato ipſius* BC, *dimidiæ Diametri* B*b*. Fig. 23.

Scilicet $DA \times AE = \overline{BC}^2$.

Ductis enim per extremitates M, S, diametrorum M*m*, S*s* ordinatis MP, SK, ad diametrum A*a*, fiat,

$CA = t$; $CB = c$; $CP = x$; $PM = y$;

Erit igitur *$\overline{CK}^2 = AP \times Pa = tt - xx$, * Art. 60.

Unde $\overline{CA}^2 - \overline{CK}^2 = tt - tt + xx = xx$,

Et * $AK \times Ka = \overline{CA}^2 - \overline{CK}^2 = xx$. * 5. El. 2.

Jam vero, * $\overline{BC}^2 : \overline{CA}^2 :: \overline{MP}^2 : AP \times Pa$, * Art. 48. & 63.

Hoc eſt $cc : tt :: yy : \frac{ttyy}{cc} = AP \times Pa$.

Ideoque $\overline{CK}^2 = \frac{ttyy}{cc}$; & $CK = \frac{ty}{c}$.

Porro * $\overline{CA}^2 : \overline{CB}^2 :: AK \times Ka : \overline{KS}^2$, * Art. 48. & 63.

Scilicet $tt : cc :: xx : \frac{ccxx}{tt}$. Unde $KS = \frac{cx}{t}$.

Ob * ſimilia autem triangula CPM, CAD, erit * 2. El. 6.

$CP : PM :: CA : AD$,

Vel $x : y :: t : \frac{ty}{x} = AD$.

Ita & ob ſimilia triangula CKS, CAE,

Erit CK : KS :: CA : AE,

Hoc eſt $\frac{ty}{c} : \frac{cx}{t} :: t : \frac{ccx}{ty}$. Unde AE $= \frac{ccx}{ty}$.

Ac proinde DA $\times$ AE $= \frac{ty}{x} \times \frac{ccx}{ty} = cc = \overline{BC}^2$.

COROLLARIUM.

Fig. 30. 76. Hinc per articulum 66, ſi recta DME duobus axibus A*a*, B*b*, in punctis D & E occurrens, Ellipſin tangat; & per punctum contactus M agatur Diameter M*m*; erit rectangulum ſub tangentis partibus DM, ME, æquale quadrato ſemiconjugatæ CS cum Diametro M*m*.

PROPOSITIO XII.

THEOREMA.

Fig. 30. 77. Si *tangat Ellipſin recta quælibet* ME, *axi in* E *occurrens, ac ſi a puncto contactus* M *agatur ad axem Ordinata* MP, *ut & è centro* C *recta* CS *tangenti* ME *parallela, æqualis vero ſemidiametro Conjugatæ cum diametro* M*m, quæ per punctum contactûs* M *ducitur; dico quadratum tangentis* ME *eſſe ad quadratum ſemidiametri* SC *eidem parallelæ, ſicut intercepta* EP *inter Ordinatam* MP *& punctum concursûs* E, *ad interceptam* PC *inter eandem Ordinatam* MP *& centrum* C.

Scilicet $\overline{ME}^2 : \overline{CS}^2 ::$ EP : PC.

Producta enim tangens EM, axi B*b*, conjugato cum

* Art. 76. A*a*, occurrat in D; & * erit DM $\times$ ME $= \overline{CS}^2$.

* 1. El. 6. Sed * $\overline{EM}^2 :$ DM $\times$ ME :: EM : MD.

El. 6. Unde $\overline{EM}^2 : \overline{CS}^2$ :: EM : MD :: * EP : PC.

LIB.

LIBER TERTIUS

DE

HYPERBOLA

DEFINITIONES.

I.

Fig. 51.

SINT F & *f* puncta in plano aliquo fixa, & *f*MO Regula ſatis longa, cujus altera extremitas *f* puncto *f* ea lege applicetur, ut circa *f* tanquam centrum liberè rotari poſſit: jam ſi ad alteram Regulæ extremitatem O, & ad alterum punctum F, annectatur filum OMF, (quod regulâ vel longius vel brevius eſſe debet) & circumagatur Regula circa centrum *f*, interea dum filum OMF, ope ſtyli vel paxilli M, continuo tendatur, & pars ejus MO Regulæ arctè juncta & velut agglutinata teneatur, curva AX paxilli motu deſcripta, appellatur Hyperbolæ Portio.

Conversa Regula ad alteram partem puncti F, reliqua Hyperbolæ portio ſimili prorſus motu deſcribetur.

Sed ſi, manente eâdem & Regulæ & fili longitudine, Regulæ extremitas puncto F, filum autem puncto *f* adjungatur; eadem ratione deſcribi poteſt alia curva *xaz*, priori XAZ omnino ſimilis & æqualis & ad verticem oppoſita, quæ etiam HYPERBOLA dicitur; adeo ut XAZ, *xaz*, ſimul appellentur HYPERBOLÆ OPPOSITÆ.

II.

PUNCTA fixa F, *f*, vocantur Hyperbolæ FOCI, vel UMBILICI.

III.

RECTA A*a*, quæ ad Hyperbolas oppoſitas terminatur, utrinque autem producta per focos tranſit, vocatur PRIMUS, vel PRINCIPALIS, vel TRANSVERSUS AXIS, vel LATUS TRANSVERSUM.

IV.

PUNCTUM C, ubi recta A*a* bifariam ſecatur, CENTRUM Hyperbolæ dicitur.

V.

SI per centrum C agatur ad axem A*a* perpendicularis indefinita B*b*; & a puncto A, tanquam centro, radio autem CF deſcribatur circulus ipſi B*b* in punctis B & *b* occurrens, pars B*b* iſtiuſce perpendicularis appellatur AXIS SECUNDUS.

VI.

AXES A*a*, B*b*, ſimul ſumpti CONJUGATI dicuntur, adeo ut primus A*a* ipſi B*b* ſit CONJUGATUS, & viciſſim, ſecundus B*b* ipſi A*a*.

VII. RECTÆ

VII.

Rectæ MP, MK, ex Hyperbolarum oppoſitarum punctis, M, M, ductæ ad alterutrum Axium conjugatorum parallelæ, & ab altero terminatæ, dicuntur eſſe ad hunc alterum Ordinatæ, vel Ordinatim applicatæ, adeo ut MP ſit Ordinata ad primum axem A*a*, & MK ad ſecundum Bb.

VIII.

Tertia proportionalis ad duos Axes vocatur Parameter primi in proportione termini: ideoque ſi fiat A*a* : B*b* :: B*b* ad tertiam *p*, erit tertia illa *p* Parameter ipſius A*a*.

IX.

Omnes rectæ per centrum C ductæ, appellantur Diametri; & quæ Hyperbolis Oppoſitis occurrunt, vocantur primæ Diametri; quæ autem non occurrunt, etſi in infinitum productæ, vocantur Secundæ Diametri.

X.

Recta, quæ Hyperbolæ in uno tantum puncto occurrit, utrinque autem producta extra cadit, dicitur eſſe Tangens in illo puncto.

XI.

Illa axis vel diametri cujuſvis pars, inter verticem & Ordinatam comprehenſa, vocatur Abscissa.

SCHO-

SCHOLIUM.

Fig. 32. 78. Neceſſe eſt filum, in prima definitione adhibitum, aut longius ſit aut brevius quàm filum fMO; ſi enim ei æqualis eſſet, linea paxilli motu deſcripta ejuſmodi foret, ut omnia illius puncta a duobus focis f, F, æqualiter diſtarent, quoniam, ablatâ ex filo & Regula communi parte MO, reſiduæ MF, Mf, ſemper eſſent æquales; unde conſtat lineam hac ratione deſcriptam fore rectam indefinitam Bb, per centrum C ductam ipſi Ff normalem.

Corollarium I.

Fig. 31. 79. Liquet ex prima Definitione, ſi ex puncto quovis M in Hyperbolis oppoſitis agantur ad focos F, f, rectæ MF, Mf, earum differentiam ſemper eſſe eandem,

Scilicet $Mf - MF = MF - Mf$.

Nam $Of - OMF = Mf - MF$,

Et $OF - OMf = MF - Mf$;

Sed $Of - OMF = OF - OMf$,

Ergo $Mf - MF = MF - Mf$.

Corollarium II.

80. Cadente puncto M in A, neceſſe eſt recta MF cadat ſuper rectam AF, recta autem Mf ſuper Af; ita etiam cadente M in a, cadet Mf ſuper af, & MF ſuper aF; unde

$Mf - MF = Af - AF$,

Et $MF - Mf = aF - af$,

* Art. 79. Sed * $Mf - MF = MF - Mf$;

Ergo $Af - AF = aF - af$;
Sed $Af - AF = Ff - 2AF$,
Et $aF - af = fF - 2af$,
Ideoque $Ff - 2AF = fF - 2af$; & $AF = af$.

Hinc patet, 1*mo*.

Diſtantiam focorum Ff biſecari in centro C,
Nam * $CA = Ca$; & $AF = af$, * Def. 4.
Unde $CA + AF = Ca + af$.

2*do*. Differentiam rectarum MF, M*f*, primo axi A*a* ſemper æquari,

Nam in Hyperbola XAZ, erit
$Mf - MF = Af - AF$;
Sed $Af - AF = Af - af = Aa$,
Unde $Mf - MF = Aa$.
Et in Hyperbola oppoſita *xaz*, erit,
$MF - Mf = aF - af$,
Sed $aF - af = aF - AF = Aa$.
Unde & $MF - Mf = Aa$.

COROLLARIUM III.

81. CONSTAT ex definitione quinta, primò ſecundum axem B*b* biſariam ſecari in C, nam ob radios AB, A*b* æquales, * erit, * 47. El. 1.

$\overline{BC}^2 + \overline{CA}^2 = \overline{bC}^2 + \overline{CA}^2$, unde ablato communi quadrato $\overline{CA}^2$, evadit $\overline{BC}^2 = \overline{Cb}^2$ & $BC = Cb$.

2*do*. SI ſuper axem B*b* ſumatur pars CE ipſi CA, dimidio axis primi A*a*, æqualis; & agatur Hypothenuſa AE; ſecundum Axem B*b* primo A*a* majorem, æqualem, vel minorem eſſe, prout recta CF major, æqualis, vel minor fuerit ipſa Hypothenuſa AE;

* Def. 5. SIT enim CF major quam AE, & *erit Ab major quam AE, & $\overline{AC}^2 + \overline{Cb}^2$ majora quam $\overline{AC}^2 + \overline{CE}^2$; ablato igitur communi $\overline{AC}^2$, erit $\overline{Cb}^2$ majus quam $\overline{CE}^2$; ideoque Cb vel Bb major quam AC vel Aa. Atque ita reliqui caſus patent.

3*tio*. SI ſuper axem Aa, ex utraque parte centri C, ſumantur partes CF, Cf, ſingulæ Hypothenuſæ AB æquales; puncta F, f, eſſe binos Hyperbolæ focos, nam
* Def. 5. radius AB diſtantiæ focali ſemper *æqualis eſt.

COROLLARIUM IV.

82. UNDE ex datis duobus axibus Aa, Bb, cognito etiam Aa eſſe primum, patet methodus Hyperbolas op-
* Art. 81. poſitas deſcribendi. *Inventis enim ſuper primum axem Aa focis F, f, ad punctum F annectatur fili OMF extremitas altera F, altera autem O ad extremitatem O Regulæ OMf, quæ circa focum f, tanquam centrum,
* Art. 78. rotari poſſit, & quæ filo adhibito vel *longior vel bre-
* Art. 80. vior ſit exceſſu vel defectu, ipſi Aa *æquali. Deſcriptis igitur, ope hujus Regulæ & fili, duabus Hyperbolis oppoſitis XAZ, xaz, ſicut in prima Definitione, conſtat eas habere pro axe primo rectam Aa, pro ſecundo autem rectam Bb, ideoque eſſe Hyperbolas quæſitas.

QUO longior fuerit Regula OMf, manente focorum intervallo, eo majores erunt Hyperbolarum oppoſitarum portiones; adeo ut pro libitu augeri vel diminui poſſint portiones illæ, ſi & Regula & filum æqualiter augeantur vel diminuantur.

SED ſi, & Regula & focis non mutatis, longius tantum filum adhibeatur, Hyperbola alterius ſpeciei deſignabitur; & ſi adhuc paulo longius, adhuc alterius; donec tandem, filo Regulæ duplæ æquali exiſtente, recta linea loco Hyperbolæ rurſus deſcribatur.

MUTATA autem focorum diſtantia, & differentia inter fili & regulæ duplæ longitudinem pariter mutatâ, Hyperbolæ ejuſdem quidem ſpeciei deſcribi poſſunt, ſed quarum partes ſimiles magnitudine different.

COROLLARIUM V.

83. QUADRATUM ipſius CB, dimidii axis ſecundi, æquatur rectangulo ſub AF & F*a*, partibus axis primi A*a*, inter focum F & terminos ipſius A*a* comprehenſis,

Dico $\overline{CB}^2 = AF \times Fa$.

Sit CF vel * AB $= m$; CA vel C*a* $= t$; & * Def. 5.

Erit * $\overline{BC}^2 = \overline{AB}^2 - \overline{CA}^2 = mm - tt$, * 47. El. 1.

Sed AF $=$ CF $-$ CA $= m - t$; & F*a* $= m + t$.

Unde AF $\times$ F*a* $= mm - tt = \overline{BC}^2$.

PROPOSITIO I.

THEOREMA.

84. SI *ad primum axem* Aa *agatur Ordinata* MP, *&* *ſuper eundem axem productum ſumatur pars* AD *ipſi* MF *æqualis (ex parte ſcilicet foci* F, *quando punctum* M *cadit in Hyperbola* XAZ, *& ex parte foci* f, *quando cadit in Hyperbola* xaz) *dico eſſe* CA : CF :: CP : CD. Fig. 33. & 34.

Sit CA vel C*a* $= t$; CF vel C*f* $= m$; CP $= x$; PM $= y$; CD $= z$. Et erit in primo caſu (Fig. 33.)

AD vel MF $=$ CD $-$ CA $= z - t$.

*a*D ſive * M*f* $=$ CD $+$ C*a* $= z + t$, * Art. 80.

FP $=$ CF $-$ CP, vel CP $-$ CF $= m - x$, vel $x - m$, prout punctum P ſupra vel infra focum F cadat.

P*f* $=$ CP $+$ C*f* $= x + m$.

Porro in ſecundo caſu (Fig. 34) erit,
AD vel $MF = CD + CA = z + t$.
aD ſive $Mf = CD - Ca = z - t$.
$FP = CP + CF = x + m$.
$Pf = Cf - CP$, vel $CP - Cf = m - x$, vel $x - m$, prout punctum P infra vel ſupra focum f cadat.

Unde in utriſque caſibus (Fig. 33 & 34.) erit $MF = CD \mp CA = z \mp t$; & $PF = x \mp m$.

*47. El. 1. Sed * $\overline{MF}^2 = \overline{MP}^2 + \overline{PF}^2$.
Hoc eſt,
$zz \mp 2tz + tt = yy + xx \mp 2mx + mm$.
Porro in utriſque caſibus,
Erit $Mf = z \pm t$; & $Pf = x \pm m$.
Sed $\overline{Mf}^2 = \overline{MP}^2 + \overline{Pf}^2$.
Unde $zz \pm 2tz + tt = yy + xx \pm 2mx + mm$.
Ideoque, quantitatibus in priori æquatione reſpectivè ſubductis ex quantitatibus in poſteriori, evadit,

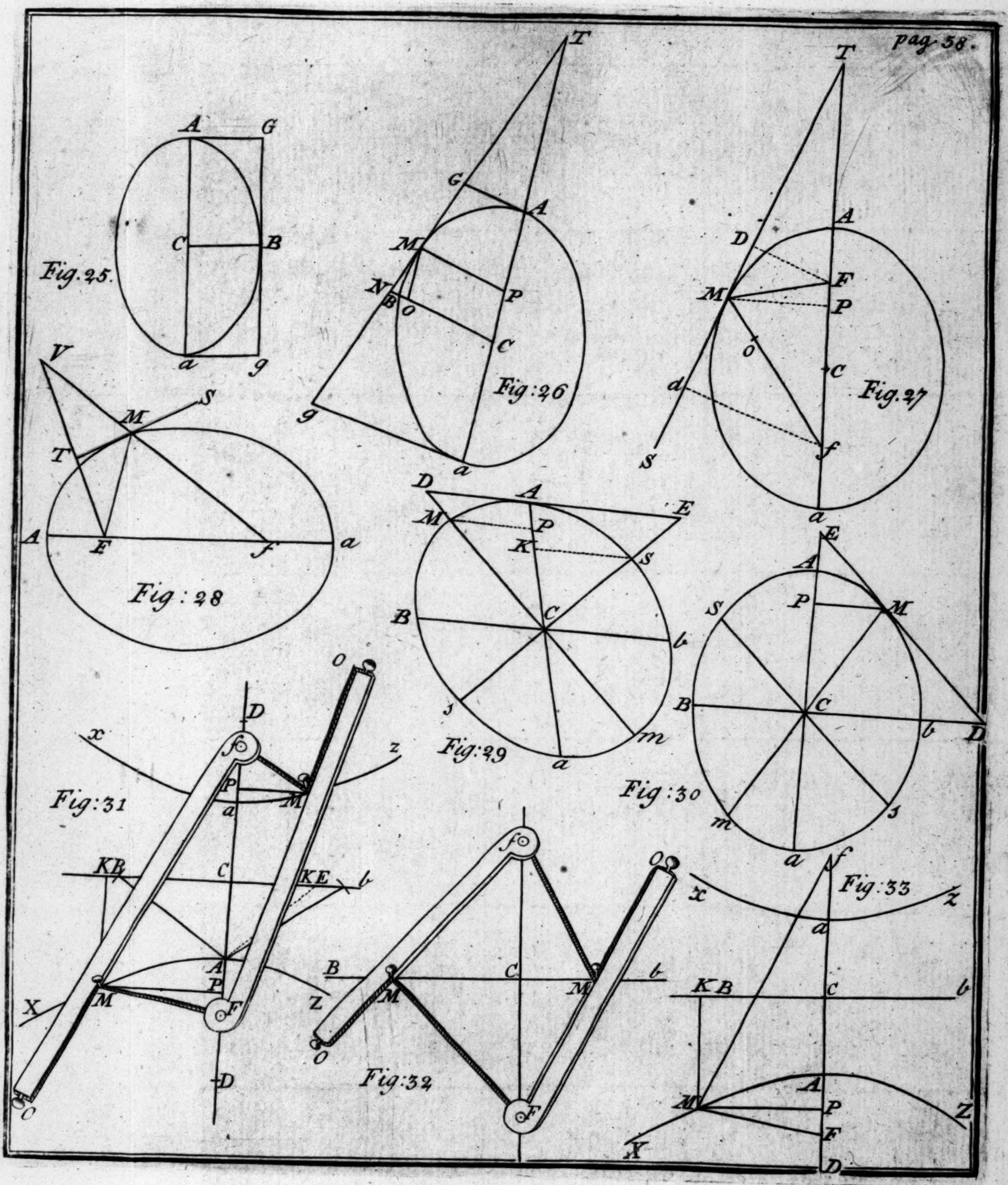
$$4tz = 4mx, \ \& \ tz = mx.$$

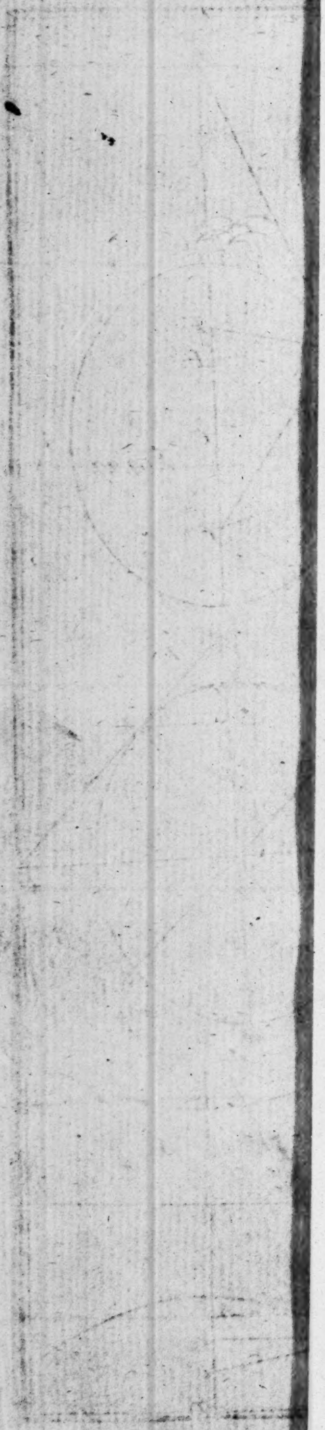
*Art. 45. Unde * $t : m :: x : z$ vel $\frac{mx}{t}$,

Hoc eſt, $CA : CF :: CP : CD = \frac{mx}{t}$.

COROLLARIUM.

Fig. 33. 85. HINC ſi fiat CA vel $Ca = t$; CF vel $Cf = m$; $CP = x$; ſemper erit in primo caſu, $MF = AD = CD - CA = \frac{mx}{t} - t$. Et $Mf = aD = CD + CA = \frac{mx}{t} + t$, puncto M cadente in Hyperbola XAZ.


pag. 58.
Fig. 25.
A G C B a g
Fig: 26
T G A M N B O P C g a
Fig: 27
T A D F M P o C d f s a
Fig: 28
V S M T A F f a
Fig: 29
D A E M P K s C B b s m a
Fig: 30
E A P M S B C b D m s a
Fig: 31
O D x z f P a M C KB KE b A P M F X Z D O
Fig: 32
f C B b M M O O F
Fig: 33
f x z a KB c b A M P F X Z D



In ſecundo autem caſu, puncto M cadente in Hyperbola *xaz*, erit MF $= \frac{mx}{t} + t$. Et M*f* $= \frac{mx}{t} - t$. Fig. 34.

PROPOSITIO II.

THEOREMA.

86. QUADRATUM *Ordinatæ cujuſvis* MP *ad Axem* A*a*, *eſt ad rectangulum ſub* AP *&* P*a illius axis producti partibus, ut quadratum conjugati* B*b ad ipſius* A*a quadratum.*

Scilicet $\overline{MP}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.

Iiſdem poſitis, quæ in præcedenti Propoſitione, ex triangulo rectangulo MPF eadem elicitur æquatio, ac prius,

Nempe $zz \mp 2tz + tt = yy + xx \mp 2mx + mm$.

Unde ſi in hac æquatione loco ipſius *z* ſubſtituatur $\frac{mx}{t}$ ipſi *z* * æqualis, oritur, * Art. 84.

$$yy + xx \mp 2mx + mm = \frac{mmxx}{tt} \mp 2mx + tt.$$

Unde $ttyy = mmxx - mmtt - tt + t^4$,

Ideoque * $yy : xx - tt :: mm - tt : tt$. * Art. 43.

Hoc eſt, $\overline{PM}^2 : AP \times Pa ::$ * $\overline{BC}^2 : \overline{CA}^2$. * Art. 83.

Sed $\overline{BC}^2 : \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$.

Ergo $\overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.

COROLLARIUM I.

87. QUADRATUM Ordinatæ cujuſvis MK ad ſecundum axem B*b*, eſt ad ſummam quadratorum ex CK,

CB, eſt quadratum conjugati A*a* ad quadratum ipſius B*b*.

Dico $\overline{MK}^2 : \overline{CK}^2 + \overline{CB}^2 :: \overline{Aa}^2 : \overline{Bb}^2$.

Sit $Bb = 2c$; eſt autem $MK = CP = x$; $CK = PM = y$.

* Art. 86. Jam verò * $\overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.
Hoc eſt, $yy : xx - tt :: 4cc : 4tt$,
* 16. El. 6. Unde * $4ccxx = 4ttyy + 4cctt$,
* Art. 43. Ideoque * $xx : yy + cc :: 4tt : 4cc$.
Hoc eſt, $\overline{MK}^2 : \overline{CK}^2 + \overline{CB}^2 :: \overline{Aa}^2 : \overline{Bb}^2$.

COROLLARIUM II ET FUNDAMENTALE.

Fig. 35 & 36. 88. SIT *Aa* primus vel ſecundus axis $= 2t$, ejus conjugatus $= 2c$, Parameter $= p$; ſingulæ ordinatæ PM $= y$; partes CP, inter centrum & ordinatam, $= x$. His poſitis, erit ſemper

* Art. 86. * $yy : xx - tt :: 4cc : 4tt$,
Scilicet $\overline{PM}^2 : \overline{CP}^2 - \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$,
* Art. 87. Sed * $\overline{PM}^2 : \overline{CP}^2 + \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$,
Ergo $\overline{PM}^2 : \overline{CP}^2 \mp \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$.

Hoc eſt, $yy : xx \mp tt :: 4cc : 4tt$ (ubi obſervare eſt, ſignum eſſe — in quantitate $xx \mp tt$, quando A*a* eſt primus axis, ſignum vero eſſe +, quando A*a* eſt ſecundus, ideoque loco ipſius $\overline{CP}^2 - \overline{CA}^2$ ſubſtitui poteſt rectangulum $AP \times Pa = xx - tt$.)

* Def. 8. Porro * $2t : 2c :: 2c : p$.
* 20. Elem. 6. Unde * $4tt : 4cc :: 2t : p$.
Ideoque $yy : xx \mp tt :: p : 2t$. Ductis

Ductis autem in ſe invicem extremis & mediis primæ & ultimæ proportionis terminis, evadit

$$yy = \frac{ccxx}{tt} \mp cc, \text{ ita \& } yy = \frac{pxx}{2t} \mp \tfrac{1}{2}pt.$$

Et cum hæc proprietas de omnibus Hyperbolarum oppoſitarum punctis vera ſit, atque eorum poſitionem quoad axes conſtanter definiat, ſequitur æquationem $yy = \frac{ccxx}{tt} \mp cc$, vel $yy = \frac{pxx}{2t} \mp \frac{1}{2}pt$ Hyperbolarum oppoſitarum naturam quoad earum axes perfectè & ad amuſſim exprimere & determinare.

COROLLARIUM III.

89. HINC quadratum Ordinatæ cujuſvis MP ad axem primum A*a*, eſt ad rectangulum AP × P*a* ſub axis producti partibus AP, P*a*, ſicut Parameter *p* ad ipſum axem A*a*.

Nam * $yy : xx - tt :: p : 2t$, * Art. 88.

Hoc eſt $\overline{MP}^2 : \overline{CP}^2 - \overline{CA}^2 :: p : Aa$.

Sed * $\overline{CP}^2 - \overline{CA}^2 = AP \times Pa$, ergo $\overline{MP}^2 : AP \times Pa :: p : Aa$. * Art. 88.

COROLLARIUM IV.

90. DUCTIS ad Axem A*a* duabus Ordinatis MP, NQ, erit

$$\overline{MP}^2 : \overline{NQ}^2 :: \overline{CP}^2 \mp \overline{CA}^2 : \overline{CQ}^2 \mp \overline{CA}^2.$$

Nam * $\overline{MP}^2 : \overline{CP}^2 \mp \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$, * Art. 88.

Et $\overline{NQ}^2 : \overline{CQ}^2 \mp \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$, Ergo $\overline{MP}^2 : \overline{CP}^2 \mp \overline{CA}^2 :: \overline{NQ}^2 : \overline{CQ}^2 \mp \overline{CA}^2$.

Notandum eſt hic etiam, rectangulum $AP \times Pa$ ſubſtitui poſſe loco ipſius $\overline{CP}^2 - \overline{CA}^2$; ita & $AQ \times Qa$ loco ipſius $\overline{CQ}^2 - \overline{CA}^2$; nam $AP \times Pa = xx - tt = \overline{CP}^2 - \overline{CA}^2$; atque eadem ratione erit $AQ \times Qa = \overline{CQ}^2 - \overline{CA}^2$.

COROLLARIUM V.

91. SI per punctum quodvis P in alterutro axium A*a* (producto ſi fuerit primus) agatur recta MPM Conjugato B*b* parallela, ea uni vel binis Hyperbolis oppoſitis occurret in punctis M & M a puncto P æquè remotis.

Cum enim rectæ MP, MP, axi B*b* ſint parallelæ, erunt etiam Ordinatis ipſis * parallelæ, ac proinde Hyperbolis occurrunt, unde & MP, MP, etiam ordinatæ; ideoque quadrata $\overline{MP}^2$, $\overline{MP}^2$ eidem quantitati $\frac{ccxx}{tt} \mp cc$ * æqualia ſunt; unde MP = MP.

* Def. 7.

* Art. 88.

COROLLARIUM VI.

Fig. 35 & 36.

92. QUONIAM $yy = \frac{ccxx}{tt} \mp cc$, conſtat quo major fuerit CP, vel x, ex alterutrâ parte centri C, eo majorem fieri PM vel y, ex alterutra parte axis A*a* ſumptam; ideoque auctâ in infinitum CP vel x, æqualiter augebitur PM vel y.

Et contra, quo minor fuerit CP vel x, eo minor erit PM vel y; adeo ut, ſi in (Fig. 35.) CP æqualis fuerit ipſi CA vel C$a = t$ (quando Aa eſt primus axis, ideoque * $yy = \frac{ccxx}{tt} - cc$) prorſus evaneſcat PM vel y. * Art. 88.

Evaneſcente autem CP vel x (quando Aa ſecundus axis fuerit) ſingula PM, quæ * ex hac Hypotheſi fit CB vel C$b = c$, minor erit omnium ex utravis parte centri C ordinatarum. Fig. 29.

Unde liquet primò, ſi (in Fig. 36) per terminos B, b, axis primi Bb agantur rectæ ſecundo axi parallelæ, eas eſſe in iſtis punctis B, b, tangentes.

2*do.* Hyperbolas oppoſitas ab axibus earum conjugatis magis magiſque in infinitum recedere, eoque remotiora eſſe Hyperbolæ cujuſvis puncta ab axe ſecundo, quo longius a primo axe diſtant; hoc tantum diſcrimine, quod axis primus binis Hyperbolis oppoſitis in uno puncto occurrat, productus autem intus cadat; quum tamen ſecundus, totus cadat intra binas Hyperbolas oppoſitas, neque iis unquam occurrit, etſi in infinitum productus.

COROLLARIUM VII.

93. SEQUITUR etiam, ex eò quod $yy = \frac{ccxx}{tt} \mp cc$, ſi puncta P & P æqualiter diſtant a centro C, ordinatas PM, PM, æquales eſſe: nam ſi CP æqualis ſit ipſi CP, erit $x = x$, ideoque quantitas $\frac{ccxx}{tt} \mp cc$ eadem erit in utroque caſu (nam quantitates cc & tt ſunt conſtantes) unde yy ex utraque centri parte idem erit quadratum; idcirco PM = PM.

Unde conſtat, ſi recta MM ad Hyperbolam unam vel ad binas terminata, biſariam ſecetur ab axe quovis Bb

in puncto K a centro C diverſo, hanc eſſe conjugato A*a* parallelam.

Actis enim axi B*b* parallelis MP, MP; recta PP
*2 Elem. 6. *biſecabitur in C, quoniam MM biſecatur in K; ideoque Ordinatæ PM, PM erunt æquales: unde recta
*33. El. 1. MM ipſi A*a* parallela *erit.

COROLLARIUM VIII.

94. SI plani Hyperbolici pars una ſuper axem A*a* ita convertatur, ut in alteram plani partem cadat, hoc eſt, ſi (in Fig. 36,) ubi axis A*a* eſt ſecundus, omnes perpendiculares, MP, BC, *&c.* ponantur ſuper omnes perpendiculares MP, *b*C, *&c.* hæ illis perfectè congruent, ideoque puncta B, M, *&c.* cadent ſuper puncta *b*, M, *&c.* & tota Hyperbola MBM ſuper totam Hyperbolam M*b*M, ex eo quod omnes illæ perpendiculares B*b*, MM,
*Art. 91. *&c.* bifariam * ſecentur in punctis C & P, *&c.*

Eadem ratione probatur portiones Hyperbolicas MAM, M*a*M eſſe etiam inter ſe æquales, quando A*a* (in Fig. 35) eſt primus Axis.

DEFINITIONES.

XII.

Fig. 37. ACTIS ex centro C duabus rectis indefinitis CG, C*g*, ad rectas AB, A*b* (per terminum A Axis primi A*a* ad terminos B, *b*, Axis ſecundi B*b* ductas) parallelis; hæ duæ rectæ vocantur ASYMPTOTI Hyperbolæ MAM; productæ autem in alteram partem centri C, dicuntur ASYMPTOTI Hyperbolæ M*a*M.

XIII. QUA-

XIII.

Quadratum partis CG vel C*g* inter centrum C & punctum G, ubi recta AB vel A*b* Aſymptoto occurrat, comprehenſæ, appellatur Dignitas Hyperbolæ MAM, vel oppoſitæ M*a*M.

Corollarium I.

95. Manifestum eſt angulum GC*g* ab Hyperbolæ
Aſymptotis factum (vel angulum BA*b* ipſi GC*g*
* æqualem) recto minorem, æqualem, aut majorem * 27. El. 1.
eſſe, prout axis ſecundus B*b* axe primo A*a* minor, æqualis, aut major fuerit.

Sit enim ſecundus axis B*b* primo axe A*a* minor, &
erit latus CB minus latere CA, & angulus CAB * minor * 18. El. 1.
angulo CBA; ſed angulus BCA eſt * rectus: unde * Def. 5.
CAB minor ſemirecto. Eodem prorſus ratiocinio oſtenditur CA*b* ſemirecto eſſe minorem; ergo totus angulus BA*b* vel GC*g* recto minor eſt. Similiter etiam reliqui caſus facillimè demonſtrantur.

Corollarium II.

96. Rectæ omnes CG, GA, GB, C*g*, *g*A, *gb*, ſibi mutuo æquantur;

Nam GC ipſi A*b* parallela * eſt, unde * Def. 12.
Erit * BG : GC :: BA : A*b*, * 4. El. 6.
Sed * BA = A*b*; ergo & BG = GC. * Def. 5.
Porro * BG : GA :: BC : C*b*; * 2. El. 6.
Sed * BC = C*b*. Ergo & BG = GA. * Art. 81.
Et * cum GA = C*g*; res patet. * 34. El. 1.

COROLLARIUM III.

97. DIGNITAS Hyperbolæ æquatur quartæ parti ſummæ quadratorum ex duobus ſemiaxibus. Scilicet $\overline{CG}^2 = \frac{\overline{CA}^2 + \overline{CB}^2}{4}$.

* Art. 96. Sit $CA = t$; $CB = c$; $CG = m$; & erit * $AB = 2m$.

* 47. El. 1. Sed * $\overline{AB}^2 = \overline{CA}^2 + \overline{CB}^2$,

Hoc eſt, $4mm = tt + cc$,

Unde $mm = \frac{tt + cc}{4}$, Vel $\overline{CG}^2 = \frac{\overline{CA}^2 + \overline{CB}^2}{4}$.

PROPOSITIO III.

THEOREMA.

Fig. 37. 98. SI *in alterutra Hyperbolarum oppoſitarum per punctum quodvis* M, *agatur ad axem* Aa *normalis* Rr, *ipſi* Aa *occurrens in* P; *dico rectangulum ſub* RM *&* Mr *æquari quadrato ipſius* BC, *dimidii axis ſecundi* Bb.

Scilicet $RM \times Mr = \overline{BC}^2$.

Sit $CA = t$; $CB = c$; $CP = x$; $PM = y$; & erunt
* Art. 15. triangula ACB, CP*r* * ſimilia; & ob eandem rationem AC*b*, CPR ſunt ſimilia;
Unde $AC : CB$ vel $Cb :: CP : Pr$ vel PR,

Vel $t : c :: x : \frac{cx}{t} = Pr$ vel PR;

Unde $RM = PR \pm PM = \frac{cx}{t} \pm y$;

$Mr = Pr \mp PM = \frac{cx}{t} \mp y$;

Ideoque $RM \times Mr = \frac{ccxx}{tt} - yy$.

Sed $*yy = \frac{ccxx}{tt} - cc$; unde ſi loco ipſius yy, in quantitate $\frac{ccxx}{tt} - yy$, ſubſtituatur $\frac{ccxx}{tt} - cc$; * Art. 88.

Erit $RM \times Mr = \frac{ccxx}{tt} - \frac{ccxx}{tt} + cc = cc = \overline{BC}^2$.

COROLLARIUM I.

99. HINC patet, omnia Hyperbolarum oppoſitarum puncta in angulis ſub Aſymptotis earum comprehenſis contineri.

Nam PR vel $Pr = \frac{cx}{t}$, & $\overline{PR}^2$ vel $\overline{Pr}^2 = \frac{ccxx}{tt}$; Sed $* \overline{PM}^2 = \frac{ccxx}{tt} - cc$; hoc eſt, $\overline{PM}^2$ minus quam $\overline{PR}^2$ vel $\overline{Pr}^2$; unde & PM minor quam PR vel Pr; & punctum M non eſt adeo remotum ab axe producto, ac puncta R & r; & cum idem de omnibus Hyperbolarum & Aſymptotôn punctis oſtendi poſſit, liquet nulla Hyperbolarum oppoſitarum puncta in angulis KCr, RCk, cadere poſſe, ſed omnia in angulis RCr, KCk, contineri. * Art. 88.

COROLLARIUM II.

100. SI in Hyperbola, vel in Hyperbolis oppoſitis, per duo quævis puncta M, N, agantur duæ rectæ Rr, Kk, primo axi Aa perpendiculares, & ab Aſymptotis

terminatæ, liquet rectangula RM × M*r* & NK × N*k* eſſe æqualia, utpote quæ eidem quadrato $\overline{BC}^2$ æquentur.
* 14. El. 6. Unde conſtat * eſſe RM : KN :: N*k* : M*r*.

PROPOSITIO IV.

THEOREMA.

101. Si *in Hyperbola, vel Hyperbolis oppoſitis per duo quævis puncta* M, N, *agantur duæ rectæ* Hh, Ll, *ſibi invicem parallelæ, & Aſymptotis occurrentes in punctis* H, h, L, l; *dico* HM × Mh = LN × Nl.

Ductis enim per puncta M, N, rectis R*r*, K*k* primo axi A*a* perpendicularibus, conſtat triangula MRH,
* Art. 15. NKL eſſe * ſimilia,

Unde RM : KN :: HM : LN;
Sed RM : KN :: N*k* : M*r*,
Ideoque HM : LN :: N*k* : M*r*.
* Art. 15. Porro triangula M*rh*, N*kl* ſunt * ſimilia,
Unde N*k* : M*r* :: N*l* : M*h*,
Ergo HM : LN :: N*l* : M*h*;
* 16. El. 6. Et * HM × M*h* = LN × N*l*.

COROLLARIUM I.

102. Si recta NL ipſi MH parallela, per centrum C tranſeat, hoc eſt, ſi MH fiat CE, patet puncta L & *l*, in unum coitura ad C; ac proinde rectangulum NL × N*l* foret quadratum $\overline{EC}^2$; unde, ſi in alterutra Hyperbolarum oppoſitarum ex puncto quovis E agatur ad centrum C recta CE, & per punctum quodvis M recta MH*h* ipſi CE parallela, Aſymptotis occurrens in punctis

H & *h*, liquet quadratum ipſius CE æquari rectangulo HM × M*h*.

COROLLARIUM II.

103. SI in alterutra Hyperbolarum oppoſitarum per punctum quodvis N ducatur recta L*l*, Aſymptotis occurrens in punctis L, *l*; Hyperbolæ autem alterutri in puncto *n*; partes LN, *ln*, iſtiuſce rectæ L*l* inter Aſymptotos & Hyperbolarum puncta comprehenſæ, erunt æquales.

Sit enim $LN = a$; $Nn = b$; $ln = c$; & erit $Nl = Nn \mp ln = b \mp c$; unde etiam $LN \times Nl = ab \mp ac$.

Sed * $LN \times Nl = HM \times Mh$; * Art. 101.
Et * $HM \times Mh = Ln \times nl$, * Art. 102.
Ergo $LN \times Nl = Ln \times nl$,
Hoc eſt $ab \mp ac = bc \mp ac$;
Unde $ab = bc$, & $a = c$, vel $LN = ln$.

COROLLARIUM III.

104. SI in conſectario præcedenti ponamus rectam N*n* ad Hyperbolas oppoſitas terminatam, per centrum C tranſire, hoc eſt, ipſam N*n* fieri primam Diametrum DE; liquet bina puncta L, *l*, coitura in centro C; ideoque foret NL ipſa EC, & *nl* ipſa CD; ſed NL ſemper eſt * æqualis ipſi *nl*; unde & EC ipſi CD; conſtat igitur primam quamvis Diametrum biſecari in centro C. * Art. 103.

COR-

COROLLARIUM IV.

105. Si duæ rectæ Mm, Nn, ſibi invicem parallelæ, & ad Hyperbolam vel Hyperbolas oppoſitas terminatæ, Aſymptoto occurrant in punctis H, L; dico rectangula $MH \times Hm$ & $NL \times Ln$ eſſe æqualia.

Productis enim, ſi opus, rectis Mm, Nn, uſque dum alteri Aſymptoto in punctis h, l, occurrant; erit primo * $MH = mh$, unde $HM \times Mh = MH \times Hm$. Sed & $NL = nl$,

* Art. 103.

Ergo $LN \times Nl = NL \times Ln$;

Sed * $HM \times Mh = LN \times Nl$,

* Art. 101.

Igitur $MH \times Hm = NL \times Ln$.

PROPOSITIO V.

THEOREMA.

Fig. 38.

106. Si *in Hyperbola vel Hyperbolis oppoſitis per bina quævis puncta* M, N, *agantur duæ rectæ* MH, NL, *ſibi ipſis parallelæ & ab Aſymptoto alterâ terminatæ; ita & ab iiſdem punctis aliæ duæ rectæ* Mh, Nl, *parallelæ, & ab Aſymptoto alterâ terminatæ; dico rectangulum* $HM \times Mh = LN \times Nl$.

Hæc propoſitio eâdem prorſus ratione demonſtratur, quâ præcedens.

COROLLARIUM I.

Fig. 39.

107. Si rectæ MH, Mh, ita & NL, Nl, duabus Aſymptotis parallelæ fuerint, conſtat parallelogramma $MHCh$, $NLCl$ (ita & triangula CHM, CLN ipſorum * dimidia) ſibi invicem æqualia;

* 41. El. 1.

Nam *$HM \times Mh = LN \times Nl$, *Art. 106.
Unde *$HM : LN :: Nl : Mh$; *16. El. 6.
Ergo, cum anguli ad C æquentur, *ideoque angu- *34. El. 1.
lus $HMh = LNl$, *erit parallelogrammum $MHCh =$ *14. El. 6.
$NLCl$.

Corollarium II.

108. Iisdem poſitis, quæ in præcedenti conſectario, manifeſtum eſt, $CH \times HM$ æquari rectangulo $CL \times LN$;
nam ex illa Hypotheſi erit *$Mh = CH$; & $Nl = CL$; *34. El. 6.
Sed cum rectangulum *$HM \times Mh = LN \times Nl$, erit *Art. 106.
quoque $CH \times HM = CL \times LN$, hoc eſt, ſi a duobus quibuſvis punctis M, N, in Hyperbola vel Hyperbolis oppoſitis, agantur binæ rectæ, MH, NL, alteri Aſymptoto parallelæ, & ab altera terminatæ, erit ſemper
$CH \times HM = CL \times LN$; ac proinde *$CH : CL ::$ *16. El. 6.
$LN : HM$.

Corollarium III.

109. Quoniam terminus A axis primi Aa punctum ſit in Hyperbola; & recta AG Aſymptoto CG occurrens in G, alteri Cg ſit parallela, ſequitur *eſſe ſemper *Art. 108.
$CH \times HM = CG \times GA$, hoc eſt, *quadrato ipſius *Art. 96.
CG; vel, per Def. 13, dignitati Hyperbolæ.

Sit igitur $CG = m$; $CH = x$; $HM = y$; & erit ſemper $CH \times HM = \overline{CG}^2$; vel $xy = mm$. Et cum hæc proprietas omnibus Hyperbolorum oppoſitarum punctis æque conveniat, atque eorum poſitionem quoad Aſymptotos conſtanter definiat, liquet æquationem $xy = mm$ Hyperbolæ naturam quoad Aſymptotos perfecte & ad amuſſim exprimere & determinare.

COROLLARIUM IV.

110. CONSTAT ex eo quod MH $= y = \frac{mm}{x}$, quo magis augeatur CH vel x, eo magis diminui HM vel y (aucto enim fractionis denominatore, tanto diminuitur ipſa fractio) adeo ut, recta CH in infinitum producta, prorſus evaneſcat HM vel y; unde patet, Hyperbolam & Aſymptoton CH ad invicem magis magiſque accedere, ita ut earum diſtantia quantitate omni data minor ſit, nunquam tamen ſibi ipſis occurrere, quia punctum occurſus non niſi ex infinitâ Aſymptoti productione habetur: nunquam autem in infinitum producitur recta.

Idem & de alterâ Aſymptoto Cg intelligendum.

COROLLARIUM V.

111. RECTÆ ſicut A*a*, per centrum C, & intra angulum ſub Aſymptotis, ex parte Hyperbolarum, comprehenſum ductæ, binis Hyperbolis oppoſitis in uno tantùm puncto A vel *a* occurrunt, productæ autem intra Hyperbolas cadunt.

Ob angulos enim GCA, *g*CA, atque his ad verticem oppoſitos, patet rectam A*a* magis magiſque ab Aſymptotis recedere; Hyperbolæ autem oppoſitæ ad eas continuò *accedunt. * Art. 110.

Eæ etiam rectæ, quæ, ſicut B*b*, intra angulos, qui ſunt deinceps, cadunt, Hyperbolis oppoſitis nunquam occurrent, etſi in infinitum producantur; nulla enim Hyperbolarum oppoſitarum puncta intra eos angulos cadere *poſſunt. * Art. 99.

Unde perſpicuum eſt, *primas Diametros intra angulos * Def. 9.

los ab Aſymptotis factos contineri, ſecundas autem intra angulos, qui ſunt deinceps.

COROLLARIUM VI.

Fig. 40. 112. Si per punctum quodvis H in altera Aſymptotôn CE, agatur recta HM alteri C*e* parallela, ea Hyperbolæ in uno tantum puncto M occurret, diſtantia enim ejus ab Aſymptoto C*e* eadem ſemper manet, Hyperbola autem ad C*e* continuo accedit.

COROLLARIUM VII.

113. HINC, ſi in Hyperbola MN per punctum quodvis M ducantur rectæ indefinitæ MH, M*h*, ad Aſymptotos CE, C*e* parallelæ, patet,

1*mo*. Omnia Hyperbolæ oppoſitæ puncta in angulo HM*h* contineri, omnia enim intra angulum ab Aſymptotis factum continentur;

2*do*. Binas Hyperbolæ portiones intra angulos *h*MK, HM*k*, ex utraque parte anguli HM*h* cadere, neque ulla ejus puncta in angulo KM*k*, ipſi HM*h* verticali, inveniri poſſe.

3*tio*. Rectas omnes, ſicut MF, in angulo HM*h* incluſas, & verſus partem F productas, Hyperbolæ oppoſitæ occurrere in puncto N, & intra cadere; hæ enim rectæ ab ipſis MH, M*h* (ac proinde ab Aſymptotis, quæ rectis MH, M*h* parallelæ ſunt) magis magiſque recedunt.

Quòd ſi ad alteram partem puncti M producantur rectæ, ſicut MF, intra Hyperbolam MN cadent; neque ei in alio puncto præter M occurrent.

4*to*. Rectas, ſicut E*e*, in angulis, qui ipſi HM*h* deinceps ſunt, cadentes, occurrere Aſymptotis Hyperbolæ per M ductæ: ideoque cadentibus his rectis intra aliquam Hyperbolæ portionem, illæ neceſſario occurrent iſti portioni in puncto aliquo N, neceſſe eſt enim Aſymptoto extra hanc portionem poſito alicubi occurrat.

COROLLARIUM VIII.

114. Si per Hyperbolæ punctum quodvis M ducatur recta F*f* uni Aſymptotôn in puncto F, Hyperbolæ autem oppoſitæ Aſymptoto in puncto *f* occurrens, producaturque ad N, ita ut *f*N ipſi FM æqualis ſit; dico punctum N eſſe in Hyperbola oppoſita; recta enim F*f* in angulo HM*h* cadit, ideoque oppoſitæ Hyperbolæ in * Art. 103. puncto aliquo occurrit: punctum igitur N * eſt punctum occurſûs.

Porro, ſi per quodvis Hyperbolæ punctum M agatur recta E*e* ad Aſymptotos terminata; & ſuper E*e* capiatur pars *e*N ipſi EM æqualis, dico punctum N eſſe in Hyperbola.

Ducta enim recta MH uni Aſymptoto C*e* parallela, & ad alteram CH terminata, ſumatur ſuper hanc alteram pars CL ipſi HE æqualis, & ſit LN ipſi HM parallela: * Art. 112. conſtat igitur rectam LN Hyperbolæ * occurrere in puncto * Art. 108. aliquo N; & * erit CL vel HE : HM :: CH vel EL : LN. Sed & punctum N in recta E*e* eſt ejuſmo- * 2. El. 6. di, ut ſit * *e*N = EM, ex eò quòd ſit EH = CL. Ergo punctum N eſt & in Hyperbola & in recta E*e*.

PROPOSITIO VI.

THEOREMA.

Fig. 40. 115. Si *per quodvis Hyperbolæ punctum* M *agatur recta* MH *alteri Aſymptoto* Ce *parallela, &* *ad alteram* CH *termi-*

CH *terminata in puncto* H; *ſumatur autem ſuper hanc alteram pars* HD *ipſi* CH *æqualis,* & *a puncto* D *per* M *ducatur recta* DMd *Aſymptoto* Ce *occurrens in* d, *dico rectam* DMd *Hyperbolam tangere in puncto* M.

Non enim; ſed ſi fieri poteſt Hyperbolæ rurſus oc-
currat in O, & * erit $DM = Od$. Sed ob ſimilia * Art. 95.
triangula DC*d*, DHM, erit $DH : HC :: DM : Md$. Unde
cum DH * æqualis ſit ipſi HC, erit $DM = Md$, id- * Hyp.
eoque $Md = Od$, quod eſt abſurdum: non igitur re-
cta DM*d* Hyperbolæ rurſus in O, ſed in unico puncto
M occurrit.

COROLLARIUM I.

116. HINC, ſi recta DM*d* Hyperbolam tangat in M, partes DM, M*d*, erunt æquales; & vice versâ, ſi partes DM, M*d* fuerint æquales, erit DM*d* tangens in puncto M.

Unde liquet unam ſolummodo rectam DM*d*, ad Aſymptotos terminatam, Hyperbolam tangere poſſe in eodem puncto M.

COROLLARIUM II.

117. SI per punctum M, ubi recta DM*d*, ad A- Fig. 41.
ſymptotos CL, C*l*, terminata, Hyperbolam tangit, du-
catur prima Diameter MC*m*, Hyperbolæ oppoſitæ oc-
currens in puncto *m*, & per *m* agatur E*e* tangenti DM*d*
parallela; dico hanc rectam eſſe tangentem in puncto *m*.

Triangula enim CMD, C*m*E, ſunt * æqualia, & ſi- * 26. El. 1.
milia (ob latus CM ipſi C*m* * æquale, & ob angulos * Art. 104.
ad C, ita & ad D & E æquales) unde $DM = mE$. Eo-

dem prorſus modo oſtenditur $Md = me$; unde Ee bifariam ſecatur in m, quia * Dd biſecatur in M; recta igitur Eme Hyperbolam * tangit in puncto m.

* Art. 116.
* Art. 116.

Unde conſtat tangentes Dd, Ee per terminos M, m, primæ Diametri cujuſvis Mm ductas ſibi invicem eſſe parallelas, & æquales, ſi modo ad Aſymptotos terminatæ fuerint.

SCHOLIUM.

Fig. 40. 118. OSTENSUM eſt (Art. 110) eo minorem eſſe HM, quo major fuerit CH: adeo ut recta CH infinitè aucta, neceſſe ſit HM infinitè diminuatur vel evaneſcat: quod ſi CH infinitè augeatur, HD ipſi CH æqualis infinitè etiam augebitur, unde MD, HD, (ſi modo ſibi invicem non occurrant, niſi in infinitum productæ) pro parallelis haberi poſſunt, ideoque in ſe mutuo cadent, quia ex infinita productione puncta M & H in unum coeunt; hoc eſt, Aſymptoto CE, & Hyperbola ipſa in infinitum productis, Aſymptotos CE pro tangente habenda eſt, & ejus terminus infinitè diſtans pro puncto contactûs. Idem dicendum de Aſymptoto Ce; unde conſtat binas Aſymptotos pro rectis infinitis habendas eſſe, quæ Hyperbolas oppoſitas in earum terminis tangant.

DEFINITIONES.

XIV.

Fig. 41. SINT rectæ Mm, Ss, duæ Diametri, quarum una Ss tangentibus per terminos alteræ ductis ſit parallela, & a rectis MS, Ms, ab extremitate M Diametri Mm ductis, & ad Aſymptotos parallelis terminata; hæ duæ Diametri ſimul appellantur CONJUGATÆ.

Fig: 34

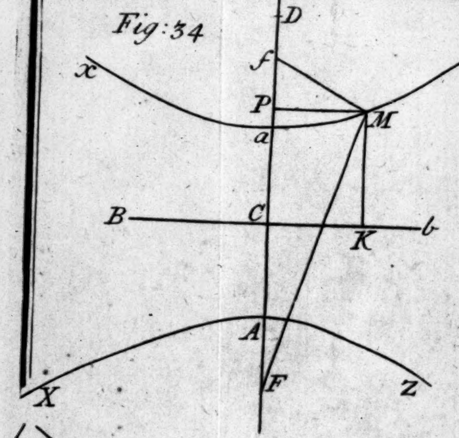


Fig: 35

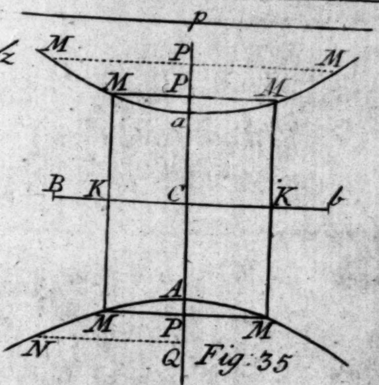


Fig: 36

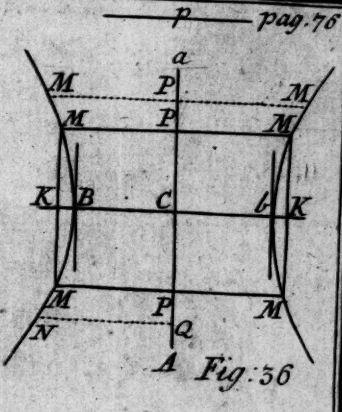


Fig: 37

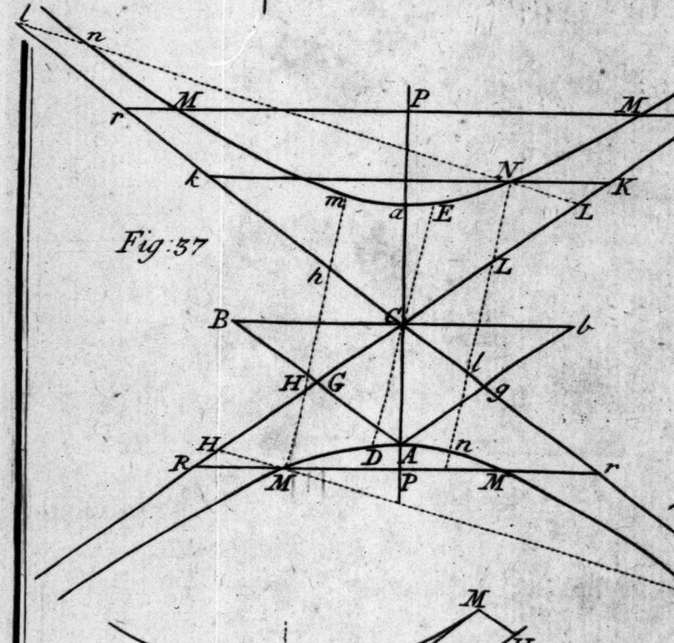


Fig: 38

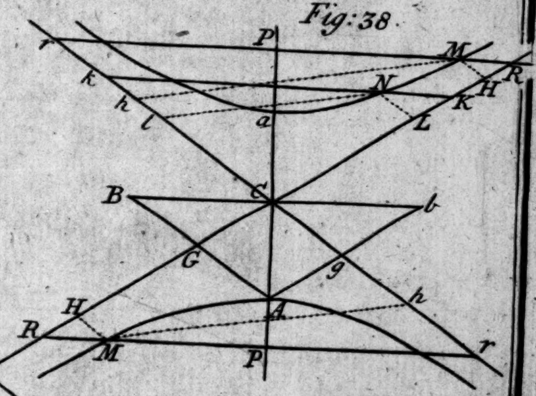


Fig: 39

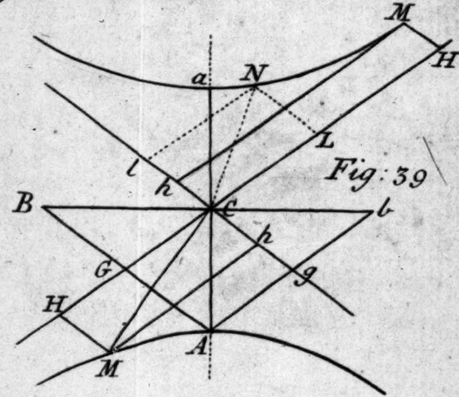
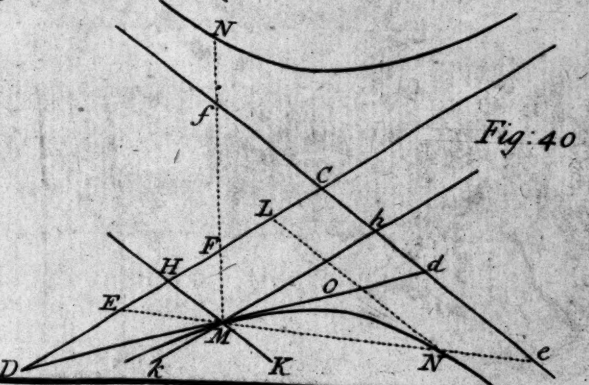


Fig: 40



XV.

RECTÆ omnes ab Hyperbolæ punctis ductæ ad unam ex Diametris Conjugatis parallelæ, & ab altera terminatæ, vocantur ORDINATÆ ad hanc alteram. Ita NO dicitur ORDINATA ad Diametrum M*m*.

XVI.

TERTIA proportionalis ad binas Diametros Conjugatas vocatur PARAMETER primi in proportione termini.

COROLLARIUM I.

119. DEFINITIO 14ta. duobus Axibus convenit; nam ſecundus Axis tangenti per extremitatem primi ductæ *parallelus eſt: & a duabus rectis ab una primi Axis ex- * Art. 92.
tremitate ductis ad Aſymptotos parallelis *terminatur. * Def. 12.
Unde conſtat duos Axes haberi poſſe pro duabus Diametris, quæ ſecum rectos conſtituant angulos.

COROLLARIUM II.

120. QUONIAM Diameter SC*s* tangenti DM*d* per terminum M Diametri MC*m* ductæ *parallela ſit, & * Def. 14.
cum hæc tangens binis Hyperbolæ per M ductæ Aſymptotis CD, C*d*, occurrat in punctis D & *d*; ſequitur hanc Diametrum SC*s* inter angulos cadere, qui ſunt deinceps angulo DC*d* ab Aſymptotis facto; ac proinde * eſſe * Art. 111.
ſecundam Diametrum.

Unde liquet ex duabus Diametris Conjugatis MC*m*, SC*s*, ſemper eſſe primam M*m*, & ſecundam S*s*.

COROLLARIUM III.

121. SECUNDA Diameter SC*s* biſecatur in centro C, & tangenti DM*d* per terminum M primæ Diametri M*m* ductæ æqualis eſt.

* 34. El. 1. Ob parallelas enim MS, C*d*, & M*s*, CD, erit * MD
= C*s*; & M*d* = CS. Unde tota DM*d* toti SC*s* æqua-
* Art. 116. lis: ſed DM*d* * biſecatur in M, ergo & SC*s* in centro C.

COROLLARIUM IV.

122. DATIS duabus Diametris Conjugatis M*m*, S*s*, cognitâ etiam prima, facile inveniuntur Aſymptoti CD, C*d*; agantur ſcilicet per centrum C rectæ CD, C*d*, ipſis MS, M*s*, per extremitatem M primæ Diametri M*m* ductis parallelæ, & erit factum.

Et vice versâ, datis duabus Hyperbolæ Aſymptotis CD, C*d*, & puncto M, facile innoteſcunt binæ Diametri Conjugatæ MC*m*, SC*s*; agatur ſcilicet MH uni Aſymptotôn C*d* parallela, alteri CD occurrens in H, & producatur MH, ita ut ſit MH = HS, & jungantur CM, CS; fiat enim MD ipſi CS parallela, & erunt triangula CHS, MHD ſimilia, ac proinde erit HM : HD
* Hyp. :: HS : HC. Sed * HM = HS, ergo & HD = HC.
* Art. 115. Erit igitur recta DM*d* * tangens in puncto M, & rectæ
* Def. 14. CM, CS * erunt ſemidiametri Conjugatæ.

COROLLARIUM V.

123. DATO ſecundæ Diametri SC*s* ſitu, facile innoteſcet ejus magnitudo, ita & prima Diameter MC*m* ipſi SC*s* conjugata.

Ducatur in angulo ab Aſymptotis facto recta L*l* ad Aſymptotos terminata & ipſi SC*s* parallela: bifariam ſecetur L*l* in puncto O, & agatur prima Diameter CO, Hyperbolæ occurrens in M; ductis igitur per punctum M rectis MS, M*s*, ad Aſymptotos C*d*, CD, parallelis, patet puncta S, *s*, ubi hæ parallelæ Diametro SC*s*, occurrunt, magnitudinem ejus definire, & primam Diametrum MC*m* eidem eſſe conjugatam; ductâ enim per punctum M rectâ DM*d* ipſi L*l* parallelâ, & ad Aſymptotos terminatâ, ea bifariam * ſecabitur in puncto M, quia L*l* biſecatur in O, ideoque D*d* * tangens eſt in puncto M.

* 2. El. 6.
* Art. 115.

Unde liquido conſtat, ſi detur ſecundæ Diametri ſitus, ejus magnitudinem eſſe certam & definitam, unamque ſolummodo invenire poſſe; ita & primæ Diametri M*m*, ipſi SC*s* conjugatæ, magnitudinem & ſitum determinari.

COROLLARIUM VI.

124. DATO ſecundæ Diametri SC*s*, & ejus Ordinatarum ſitu, data etiam magnitudine ejus, unà cum parametro, facile innoteſcunt ſitus & magnitudo primæ Diametri MC*m* ipſi SC*s* conjugatæ, una cum ipſius parametro.

Ductâ enim per centrum C indefinitâ rectâ M*m*, Diametri S*s* Ordinatis parallelâ, ſumantur ſuper rectam M*m* duo puncta M & *m* a centro C æque remota, ita ut M*m* ſit media proportionalis inter ſecundam Diametrum S*s* & parametrum ejus: inventâ deinde tertiâ proportionali ad binas rectas M*m*, S*s*, * conſtat M*m* eſſe primam Diametrum ipſi S*s* conjugatam, & habere pro ejus parametro iſtam tertiam proportionalem.

* Def. 15. & 16.

PROPOSITIO VII.

THEOREMA.

Fig. 41. 125. QUADRATUM *Ordinatæ cujuſvis* ON *ad Diametrum* Mm, *eſt ad rectangulum ſub* MO *&* Om *illius Diametri ſegmentis, ut quadratum Conjugatæ* Ss *ad quadratum ipſius* Mm.

Dico $\overline{ON}^2 : MO \times Om :: \overline{Ss}^2 : \overline{Mm}^2$.

Ductâ enim per extremitatem M Diametri primæ M*m*
rectâ D*d* ad Aſymptotos terminatâ, & ſecundæ Diame-
* Def. 14. tro S*s* parallelâ, ea * erit tangens in puncto M, ideo-
* Art. 116. que * biſecabitur in M. Producatur igitur (ex utraque
* Def. 15. parte Diametri M*m*) ordinata ON, ipſi S*s* * parallela;
& Aſymptotis in punctis L & *l* occurret, a puncto O
* 2. El. 6. * æque remotis. His poſitis,

* Art. 121. Fiat CM vel C*m* $= t$; CS vel C*s*, ſive * MD vel M*d*, $= c$; CO $= x$; ON $= y$.

Jam verò, triangula CMD, COL, ſunt ſimilia,

Unde CM : MD :: CO : OL vel O*l*,

Hoc eſt $t : c :: x : \frac{cx}{t} =$ OL vel O*l*;

Idcirco $LN = LO \pm ON = \frac{cx}{t} \pm y$.

Et $Nl = lO \mp ON = \frac{cx}{t} \mp y$,

Unde $LN \times Nl = \frac{ccxx}{tt} - yy$,

* Art. 98. Sed * $LN \times Nl = DM \times Md$,

Vel $\frac{ccxx}{tt} - yy = cc$,

Ergo $ccxx - cctt = ttyy$,

Et * $yy : xx - tt :: cc : tt$, * Art. 88.

Sive $\overline{ON}^2 : MO \times Om :: \overline{CS}^2 : \overline{CM}^2 :: \overline{Ss}^2 : \overline{Mm}^2$.

Corollarium.

126. Manifestum eſt, quæcunque in Propoſitione * ſecundâ de duobus Axibus A*a*, B*b*, fuerint demonſtrata, ope hujus propoſitionis locum habere apud binas quaſvis diametros conjugatas; & cum articuli 87, 88, 89, 90, 91, 92, 93, a ſecundâ Propoſitione clarè deducantur, verique ſint, utrum angulus ACB rectus fuerit necne, ſequitur, ſi in iſtis articulis ponamus rectas A*a*, B*b*, non eſſe axes, ſed binas quaſvis diametros conjugatas, eos adhuc veros fore, demonſtratio enim eorum etiam ex hâc Hypotheſi eadem prorſus erit. * Art. 86.

PROPOSITIO VIII.

Theorema.

127. Si *binæ rectæ* DE, FG *ad Aſymptotos terminatæ, & ſe mutuo ſecantes in puncto* O, *Hyperbolam* MA *in punctis* M *&* A *tangant; dico triangulorum* CDE, CFG, *latera circa communem angulum* C *eſſe reciprocè proportionalia.* Fig. 42.

Scilicet CD : CF :: CG : CE.

Ductis enim per puncta contactuum, M, A, rectis MH, AL, Aſymptoto CG parallelis, patet triangula CDE, HDM eſſe * ſimilia, ac proinde CE : HM :: DE : DM; ſed * DE = 2DM, unde CE = 2HM. Porro DE : EM :: DC : CH; unde etiam DC = 2CH. * 2. El. 6. * Art. 116.

Triangula

Triangula autem CFG, LFA ſunt ſimilia; ergo CF = 2CL, & CG = 2AL, quia FG = 2FA.

* Art. 108. Jam verò * 2CH : 2CL :: 2LA : 2HM,
Hoc eſt CD : CF :: CG : CE.

Corollarium I.

128. Hinc triangula ipſa CDE, CFG, communem
* 15. El. 6. angulum ad C habentia, erunt * æqualia.

Corollarium II.

129. Liquet rectam DE in eadem ratione ſecari ad puncta M & O, quâ FG ad puncta A & O.
* Art. 127. Nam * CD : CF :: CG : CE,
* 17. El. 6. Unde * erit DF : FC :: GE : EC; ideoque erit FE ipſi
* 2. El. 6. DG * parallela, & triangula FOE, DOG ſimilia, unde,
Erit DO : OE :: GO : OF,
* 18. El. 5. Et * DE : EO :: GF : FO,
* 16. El. 5. Ergo * ME : MO :: AF : AO,
Hoc eſt, DM : MO :: GA : AO.

Lemma III.

130. Si *fuerint quæcunque & quotcunque magnitudines: ratio primæ ad ultimam componitur ex rationibus mediarum.*

Sint A, B, C, D, quantitates datæ, dico eſſe A ad D in ratione compoſita ex rationibus A ad B, B ad C, C ad D; nam ratio ipſius A ad D exprimitur per quotum ipſius A per D diviſæ, hoc eſt, per $\frac{A}{D}$; atque idem de cæteris rationibus obſervandum; unde ratio compoſita ex intermediis rationibus æquatur (per Def. 5. Elem. 6.) exponentibus $\frac{A}{B}$, $\frac{B}{C}$ & $\frac{C}{D}$ in ſe mutuo ductis; videlicet $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{ABC}{BCD} = \frac{A}{D}$; Ergo ratio ipſius A ad D componitur ex rationibus mediis.

PROPOSITIO IX.

THEOREMA.

131. SI *per quodvis Hyperbolæ punctum* M *agatur ad diametrum quamvis* Aa *Ordinata* MP, *& tangens* MT *iſti diametro occurrens in* T; *dico eſſe* CP : CA :: CA : CT; *neceſſe eſt autem ut puncta* P *&* T *ex eadem parte centri* C *cadant, quando* Aa *eſt prima Diameter; & è contra, ut* P *ex unâ, &* T *ex altera parte, quando* Aa *eſt ſecunda Diameter.* Fig. 43. & 44.

Si A*a* fuerit prima diameter, producatur utrinque tangens MT, donec Aſymptotis CD, CG, in punctis D & E occurrat; ita & Ordinata PM, donec Aſymptoto CD in puncto N occurrat. Porro per punctum A agatur recta AK tangenti DE parallela, & Aſymptoto CG occurrens in K; & ſit FG tangens in puncto A, ad Aſymptotos terminata, & alteri tangenti DE in O occurrens; & *erit FG ipſi MP parallela. * Def. 14. & 15.

His poſitis, erit AP ad AC in ratione compoſitâ ex EK ad EG, & ex EG ad EC.

Nam *AP : AC : FN : FC, * 2. El. 6.

Et ratio ipſius FN ad FC, *componitur ex rationibus FN ad FD, & FD ad FC. * Art. 130.

Unde ratio AP ad AC compoſita eſt ex rationibus FN ad FD & FD ad FC.

* Art. 129. Sed FN : FD :: OM : OD :: * OA : OG :: EK : EG,
Ideoque ratio AP ad AC componitur ex rationibus EK ad EG, & FD ad FC.

Sed FD : FC :: EG : EC,
* Art. 127. (Nam * CD : CF :: CG : CE,
* 17. El. 5. Unde * FD : FC :: EG : EC.)
Ac proinde ratio AP ad AC componitur ex EK ad EG, & ex EG ad EC;

* Art. 130. Sed EK ad EC * componitur etiam ex rationibus EK ad EG, & ex EG ad EC.

Unde AP : AC :: EK : EC :: AT : TC,
Ergo AP : AC :: AT : TC,
* 18. El. 5. Ac proinde * CP : CA :: CA : CT. quod erat primum.

Fig. 44. *2do.* Jam ſit A*a* ſecunda Diameter; & agatur per centrum C recta CK Ordinatæ PM parallela, quæ Hyperbolæ in puncto B, tangenti autem MT in puncto R occurrat, & ſit MK ipſi A*a* parallela. Conſtat igitur CB
* Def. 14. * eſſe primam ſemidiametrum ipſi A*a* conjugatam, &
* Def. 15. MK * eſſe ordinatam ad ipſam CB.

His poſitis, ſit CA vel C*a* $= t$; CB $= c$; CP vel MK $= x$; PM vel CK $= y$;

Et erit CK : CB :: CB : CR, per primam partem,

Hoc eſt, $y : c :: c : \frac{cc}{y} = CR$,

Unde $RK = CK - CR = \frac{yy - cc}{y}$;

* 4. El. 6. Triangula autem KRM, CRT ſunt * ſimilia,
Unde KR : RC :: MK : CT,

Vel $\frac{yy-cc}{y} : \frac{cc}{y} :: x : \frac{ccx}{yy-cc} =$ CT;

Sed *$yy = \frac{ccxx}{tt} + cc$; ideoque $yy - cc = \frac{ccxx}{tt}$, unde ſi quantitas ccx dividatur per $\frac{ccxx}{tt}$ loco ipſius $yy - cc$, evadet, * Art. 88. & 126.

$$\frac{ccx}{yy-cc} = \frac{ccxtt}{ccxx} = \frac{tt}{x}\text{; unde CT} = \frac{tt}{x};$$

Sed * $x : t :: t : \frac{tt}{x}$, vel CP : CA :: CA : CT, quod erat ſecundum. * 16. El. 6.

PROPOSITIO X.

THEOREMA.

132. SI *in Hyperbola, cujus centrum* C, *agatur ad alterutrum axem* Aa *Ordinata* MP; *ita & recta* MG, *tangenti* MT *per* M *ductæ perpendicularis; dico eſſe ſemper* CP *ad* PG *in ratione datâ ipſius axis* Aa *ad parametrum ejus* p. Fig. 45. & 46.

Scilicet CP : PG :: Aa : p.

Sit CA vel C$a = t$; CP $= x$; PM $= y$; & erit * CT $= \frac{tt}{x}$; unde TP $=$ CP $\mp$ CT $= x \mp \frac{tt}{x} = \frac{xx \mp tt}{x}$, prout axis A$a$ primus vel ſecundus fuerit. * Art. 131.

Jam vero triangula TPM, MPG ſunt * ſimilia, unde, * 8. El. 6.

TP : PM :: PM : PG,

Vel $\frac{xx \mp tt}{x} : y :: y : \frac{xyy}{xx \mp tt} =$ PG,

Sed * $x : \frac{xyy}{xx \mp tt} :: xx \mp tt : yy$, * 16. El. 6.

Et

* Art. 88. Et * $xx \mp tt : yy :: 2t : p$,

Ergo, $x : \frac{xyy}{xx \mp tt} :: 2t : p$, hoc eſt CP : PG :: A*a* : *p*.

COROLLARIUM.

133. HINC ſi CA vel C*a* dicatur *t*; CP, *x*; PM, *y*; Parameter, *p*; erit $\overline{PM}^2 : TP \times PC :: p : Aa$.

* Art. 132. Nam * $TP \times PC = xx \mp tt$;

* Art. 88. Sed * $yy : xx \mp tt :: p : 2t$; vel $\overline{PM}^2 : TP \times PC :: p : Aa$.

PROPOSITIO XI.

THEOREMA.

Fig. 47. 134. SI *in Hyperbola vel Hyperbolis oppoſitis, ab extremitatibus* A, a, *Diametri cujuſvis* Aa *agantur rectæ* AG, ag, *Ordinatis parallelæ, & alia quæpiam recta* MT *quomodocunque contingens ducatur, dico rectangulum ſub rectis* AG, ag, *æquari quadrato dimidiæ Diametri* CB, *ſemiconjugatæ cum* Aa.

Producta MG diametro A*a* occurrat in puncto T; ducanturque per punctum M rectæ MP, MO, ipſis AG, A*a*, parallelæ.

* Art. 131. Quoniam igitur * TC : CA :: CA : CP,

Erit TC : CA :: TA : AP ; ſed CA = C*a*,

Ergo C*a* : TC :: AP : AT,

Et T*a* : TC :: TP : AT,

* 4. El. 6. Unde * *ag* : CN :: PM : AG,

Et AG × *ag* = PM × CN,

* Art. 131. Sed PM × CN = CN × CO = * $\overline{CB}^2$;

Ergo AG × *ag* = $\overline{CB}^2$.

PRO-

PROPOSITIO XII.

THEOREMA.

135. SI *ex quovis Hyperbolæ puncto* M *agantur ad focos* F, f, *rectæ* MF, Mf; *dico tangentem per punctum illud* M *ductam bifariam ſecare angulum* FMf. Fig. 48.

Scilicet $FMD = fMd$.

Ductis enim FD, fd tangenti MT perpendicularibus, ſit Aa axis primus, ipſi MT occurrens in T, & ſit MP ordinata ad axem.

Sit CA vel $Ca = t$; CF vel $Cf = m$; $CP = x$;

Et erit $*MF = \frac{mx}{t} - t$, & $Mf = \frac{mx}{t} + t$, * Art. 85.

Sed $* \frac{mx}{t} - t : \frac{mx}{t} + t :: m - \frac{tt}{x} : m + \frac{tt}{x}$, * 16 El. 6.

Hoc eſt, $MF : Mf :: CF - *CT : Cf + CT$, * Art. 131.

Vel $MF : Mf :: TF : Tf :: FD : fd$,

Ergo $MF : Mf :: FD : fd$.

ideoque triangula FMD, fMd ſunt * ſimilia, ac proinde anguli FMD, fMd, lateribus homologis DF, df, oppoſiti, ſunt æquales. * 7. El. 6.

COROLLARIUM I.

136. HINC ſi ab Hyperbolæ puncto quovis M, agantur ad focos F, f, rectæ MF, Mf, & per idem punctum M recta MT, ita ut angulus FMT æqualis ſit angulo fMT, erit recta MT tangens in puncto M.

COR-

COROLLARIUM II.

Fig. 49. 137. SI ab Hyperbolæ focis F, *f*, ad punctum quodvis tertium V inflectantur duæ rectæ FV, *f*V, quarum una *f*V axi majori A*a* æqualis ſit, altera FV a perpendiculo MS in ſe demiſſo biſecetur in S; perpendiculum illud MS Hyperbolam tanget in M.

Secet enim perpendiculum MS rectam *f*V productam in puncto M, & jungatur FM. Et erit * $VM = MF$; & angulus $SMV = SMF$. Ergo $Mf - MF = Mf - MV = Vf = Aa$, axi majori; ideoque punctum M * eſt in Hyperbola; & cum angulus $SMF = SMf$, recta SM Hyperbolam tangit.

* 4. El. 1.

* Art. 80.

Et contra, ſi perpendiculum SM Hyperbolam tangit, erit *f*V axi majori A*a* æqualis, nam ob æqualia triangula MSF, MSV, erit $MF = MV$, ideoque $Mf - MV = Mf - MF =$ *A*a*.

* Art. 80.

PROPOSITIO XIII.

THEOREMA.

Fig. 50. 138. DIFFERENTIA *quadratorum ex duabus Diametris quibuſvis Conjugatis* Mm, Ss, *æquatur differentiæ quadratorum ex duobus axibus* Aa, Bb,

$$\text{Dico } \overline{CS}^2 - \overline{CM}^2 = \overline{CB}^2 - \overline{CA}^2,$$

$$\text{Vel } \overline{CM}^2 - \overline{CS}^2 = \overline{CA}^2 - \overline{CB}^2,$$

Actis enim rectis MS, AB, ex Aſymptoto alteri C*g* erunt parallelæ, & ab altera CG bifariam ſectæ in punctis H & G; nam ob parallelas M*s*, C*h*, erit SC : C*s* :: SH : HM.

Sed *$SC = Cs$, ergo & $SH = HM$. Eâdem ratione * Art. 121.
oſtenditur AB biſecari in puncto G; unde ſi ex punctis A, M, B, S, demittantur ſuper Aſymptoton CG normales AF, ML, BE, SK; triangula GAF, GBF, ita & HML, HSK, erunt ſimilia, & *æqualia, ob latus $AG =$ * 26. El. 1.
GB, & $SH = HM$. His poſitis, fiat CG vel *$GA =$ * Art. 96.
m; GE vel $GF = a$; AF vel $BE = b$; $CH = x$; $HM = y$; & erit $CE = m + a$; $CF = m - a$.

Unde $\overline{CE}^2 + \overline{EB}^2 = \overline{CB}^2 = mm + 2am + aa + bb$,

Et $\overline{CF}^2 + \overline{FA}^2 = \overline{CA}^2 = mm - 2am + aa + bb$,

Ergo $\overline{CB}^2 - \overline{CA}^2 = 4am$.

Triangula autem GAF, HML ſunt *ſimilia, * Art. 15.
Ideoque $GA : AF :: HM : ML$,

Scilicet $m : b :: y : \frac{by}{m} = ML$ vel KS;

Sed & $GA : GF :: HM : HL$,

Hoc eſt, $m : a :: y : \frac{ay}{m} = HL$ vel HK,

ac proinde $CK = CH + HK = x + \frac{ay}{m}$,

Et $CL = CH - HL = x - \frac{ay}{m}$,

Ergo $\overline{CS}^2 = \overline{CK}^2 + \overline{KS}^2 = xx + \frac{2axy}{m} + \frac{aayy}{mm} + \frac{bbyy}{mm}$,

Et $\overline{CM}^2 = \overline{CL}^2 + \overline{LM}^2 = xx - \frac{2axy}{m} + \frac{aayy}{mm} + \frac{bbyy}{mm}$,

Unde $\overline{CS}^2 - \overline{CM}^2 = \frac{4axy}{m}$,

Sed *$xy = mm$; ac proinde ſubſtitutione factâ, erit * Art. 109.
$\frac{4axy}{m} = 4am$, hoc eſt, $\overline{CS}^2 - \overline{CM}^2 = 4am$, ſed & $\overline{CB}^2 - \overline{CA}^2 = 4am$, ideoque $\overline{CS}^2 - \overline{CM}^2 = \overline{CB}^2 - \overline{CA}^2$.

Si angulus GC*g* ab Aſymptotis factus eſſet acutus (etſi quidem in hac figura ſit obtuſus, atque ex hac hypotheſi inſtituitur demonſtratio) conſtat rectam CF majorem fore, quam CE; angulus enim AGE ſemper * 34. El. 1. * æqualis eſt angulo GC*g*, & in hoc caſu eſſet acutus; ſed ſi perpendicularis AF inter puncta G & C caderet, * 16. El. 1. angulus acutus AGE major * eſſet angulo recto, quod fieri nequit; unde recta AF infra G ex hac Hypotheſi cadat neceſſe eſt; atque ita facilè oſtenditur rectam BE ſupra G cadere, ex eo quod BGC tum eſſet acutus. Et in hoc caſu eâdem prorſus ratione oſtendi poſſet, eſſe

$$\overline{CM}^2 - \overline{CS}^2 = \overline{CA}^2 - \overline{CB}^2;$$

Sin vero angulus GC*g* ab Aſymptotis factus fuerit rectus; AGE, iſti angulo ſemper æqualis, eſſet quoque rectus; ſed & ex hypotheſi anguli CHM, CHS, eſſent recti; unde triangula CHM, CHS, ita & CGA, CGB, forent æqualia, & ſemidiameter CM = CS, ita & ſemiaxis CB = CA; & cum in hoc caſu differentia inter binas diametros conjugatas M*m*, S*s*, ita & inter axes A*a*, B*b*, evaneſcat, liquido conſtat, hanc propoſitionem veram eſſe in omnibus caſibus.

Corollarium.

139. Hinc patet, primam quamvis diametrum M*m* ſecundâ S*s* minorem, æqualem vel majorem eſſe, prout angulus ab Aſymptotis factus fuerit obtuſus, rectus, vel acutus; ſit enim GC*g* obtuſus, ideoque AGH & CHS, etiam obtuſi; unde in triangulis CHS, CHM, angulus * 24. El. 1. CHS major erit angulo CHM, ideoque * latus CS latere CM majus.

DEFINITIO.

XVII.

Duæ Hyperbolæ oppoſitæ tum Æquilateræ, vel Rectangulæ dicuntur, cum binæ earum diametri conjugatæ ſibi invicem æquales fuerint, vel cum angulus ab Aſymptotis factus fuerit rectus.

Corollarium II.

140. Hinc, ſi ab Hyperbolæ Æquilateræ puncto Fig. 51. quovis M agatur ad quamvis Diametrum A*a* Ordinata MP, erit ſemper $\overline{MP}^2 = \overline{CP}^2 \mp \overline{CA}^2$; ſigno exiſtente —, quando A*a* eſt prima diameter; ſigno autem +, quando A*a* eſt ſecunda;

Nam * $\overline{MP}^2 : \overline{CP}^2 \mp \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2$. * Art. 88. & 126.
Sed in Hyperbola Æquilatera B*b* = A*a*, ideoque $\overline{MP}^2 = \overline{CP}^2 \mp \overline{CA}^2$.

DEFINITIO.

XVIII.

Sint AM, *am*, duæ Hyperbolæ oppoſitæ; recta A*a* Fig. 52. axis primus, recta autem B*b* ſecundus: ſint etiam BS, *bs*, duæ aliæ Hyperbolæ oppoſitæ, quarum è contrariò primus axis ſit B*b*, ſecundus A*a*; hæ duæ Hyperbolæ BS, *bs* dicuntur eſſe alteris duabus AM, *am*, Conjugatæ; & hæ quatuor ſimul appellantur Hyperbolæ Conjugatæ.

COROLLARIUM.

141. MANIFESTUM eſt, rectas B*a*, A*b*, ſibi invicem eſſe parallelas, ex eo quòd rectæ A*a*, B*b*, bifariam *ſecentur in puncto C; unde conſtat, Hyperbolam BS ipſi AM conjugatam * habere pro altera Aſymptoto rectam CG ipſius AM Aſymptoton; & pro alterâ, rectam C*g* alteram Hyperbolæ AM Aſymptoton verſus C indefinitè productam, hæ enim rectæ per centrum C tranſeunt, & duabus rectis B*a*, BA, per terminum B, primi axis B*b* Hyperbolæ BS ad terminos A, *a*, axis ſecundi A*a* ductis ſunt parallelæ.

* Def. 4. & 5.

* Def. 12.

Conſtat igitur rectas CG, C*g*, ipſis A*b*, AB, parallelas, & ex utrâque parte centri C indefinitè productas, non eſſe ſolummodo Aſymptotos Hyperbolarum oppoſitarum AM, *am*, verum etiam binarum BS, *bs*, ipſis AM, *am*, Conjugatarum.

PROPOSITIO XIV.

THEOREMA.

142. SI *in alterâ Aſymptotôn* CG *Hyperbolarum* AM, BS, *per punctum quodvis* H *agatur recta* MS, *alteri Aſymptoto* Cg *parallela; dico hanc rectam Hyperbolis* AM, BS, *occurrere in duobus punctis* M *&* S *a puncto* H *æquidiſtantibus.*

* Art. 112. Conſtat *enim 1*mo*, rectam MS Hyperbolis alicubi occurrere in punctis M & S.

* Art. 109. 2*do*. Ob Hyperbolam AM* erit rectangulum $CH \times HM = CG \times GA$; & ob Hyperbolam BS, erit $CH \times HS = GG \times GB$: ſed * $GA = GB$; unde $CG \times GA = CG \times GB$; ideoque $CH \times HM = CH \times HS$, & $HM = HS$.

* Art. 96.

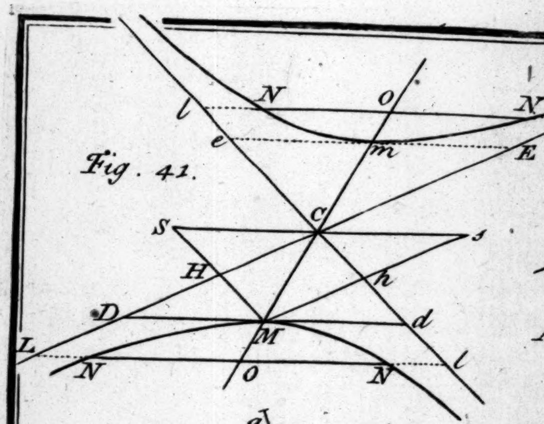
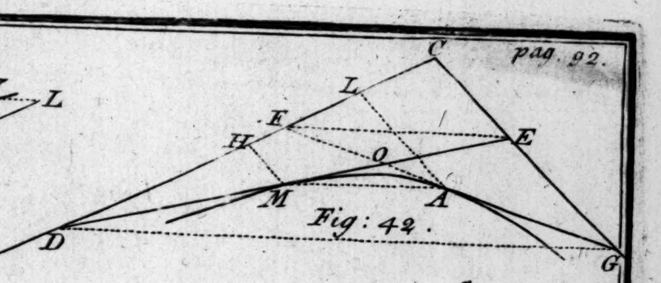


Fig. 41.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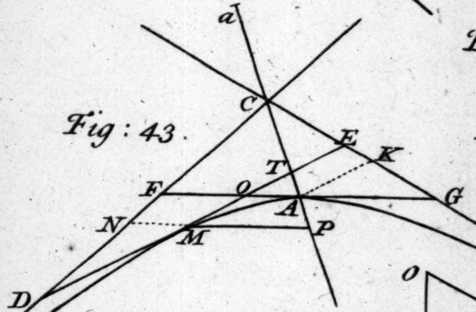
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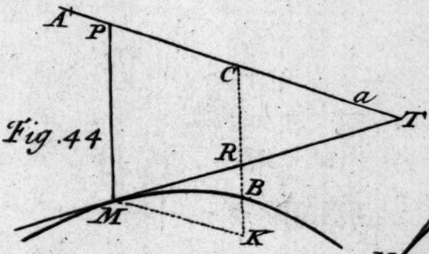
Fig. 44

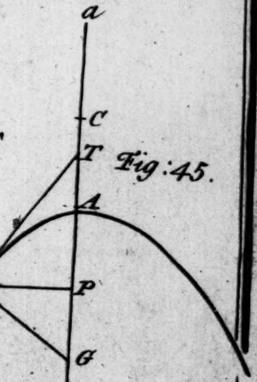
Fig: 45.

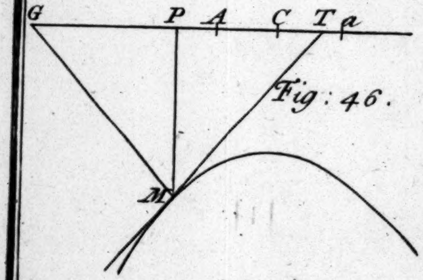
Fig: 46.

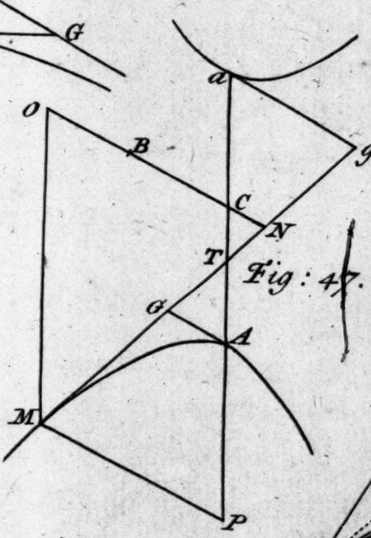
Fig: 47.

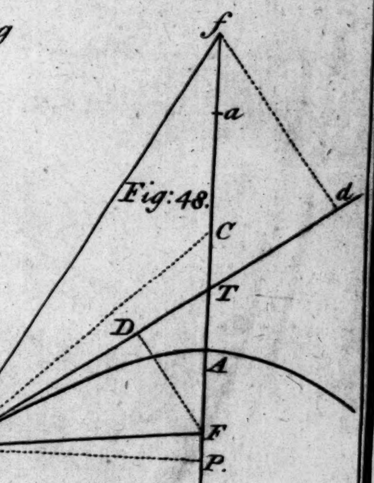
Fig: 48.

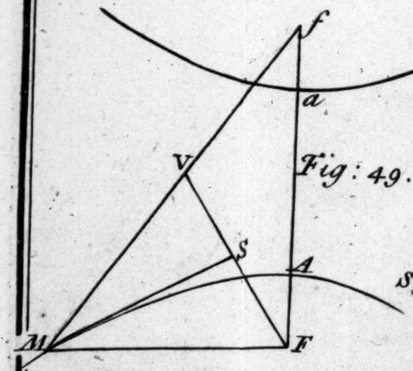
Fig: 49.

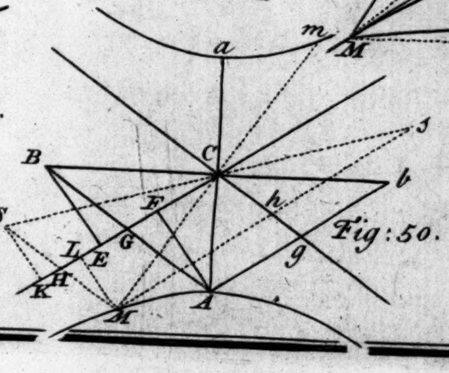
Fig: 50.

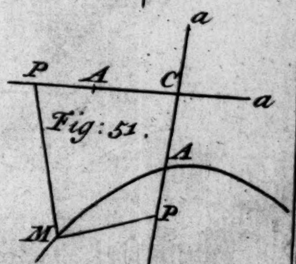
Fig: 51.

COROLLARIUM I.

143. SI in duabus Hyperbolis AM, BS, per puncta M, S, ductæ intelligantur diametri duæ MC*m*, SC*s*, ad duas alias Hyperbolas *am*, *bs*, terminatæ, liquet in Hyperbolis oppoſitis AM, *am*, diametrum S*s* eſſe ſecundam, ac Conjugatam primæ M*m*; & vice versâ, diametrum M*m* eſſe in Hyperbolis BS, *bs*, ſecundam, primæ S*s* conjugatam; unde patet,

Binas quaſvis diametros conjugatas in duabus Hyperbolis oppoſitis AM, *am*, eſſe etiam diametros conjugatas in duabus Hyperbolis BS, *bs*, ipſis AM, *am*, Conjugatis; hoc tantum diſcrimine, quòd prima diameter M*m*, quoad Hyperbolas BS, *bs*, fit ſecunda, & è contrariò ſecunda S*s*, prima.

COROLLARIUM II.

144. Unde liquido apparet, Hyperbolas BS, *bs*, ipſis AM, *am*, Conjugatas, per terminos S, *s*, omnium diametrorum SC*s*, quoad Hyperbolas AM, *am*, ſecundarum tranſire; & viciſſim Hyperbolas AM, *am*, ipſis BS, *bs*, Conjugatas per terminos M, *m*, diametrorum omnium MC*m*, quoad Hyperbolas BS, *bs*, ſecundarum, etiam tranſire.

PROPOSITIO XV.

THEOREMA.

145. SI *in Hyperbolis Conjugatis* AM, am, BS, bs, *ducantur diametri quævis Conjugatæ* Mm, Ss, *axes autem* Aa, Bb; *dico parallelogrammum ſub diametris* Mm, Ss, *contentum, æquari rectangulo ſub axibus* Aa, Bb. Fig. 53.

Per puncta M, S, agantur rectæ MD, SD, ipſis S*s*, M*m*, parallelæ, ſibi mutuo occurrentes in D; producatur MD, donec axi A*a* occurrat in T; & a puncto M demittatur MP ipſi A*a* normalis; & *erunt MD, SD, tangentes in punctis M, S; & CMDS parallelogrammum.

* Art. 92. & 126.

mum, & æquale quartæ parti parallelogrammi circa diametros Mm, Ss, deſcripti, ex eo quòd Mm, Ss, bifariam ſecentur in centro C; demittatur CE ipſi MD productæ perpendicularis, & erit rectangulum DM × CE æquale parallelogrammo CD; dico igitur eſſe DM × CE, vel CS × CE $=$ CA × CB. Sit enim CA $= t$; CP $= x$; CB $= c$; & erit,

* Art. 86. $* \overline{CA}^2 : AP \times Pa :: \overline{CB}^2 : \overline{MP}^2,$

* Art. 88. Vel $tt : * xx - tt :: cc : \frac{ccxx}{tt} - cc = \overline{MP}^2,$

Unde $\overline{CM}^2 = xx + \frac{ccxx}{tt} - cc.$

* Art. 131. Porro $* CP : CA :: CA : CT,$

Vel $x : t :: t : \frac{tt}{x} = CT,$

Unde $PT = CP - CT = x - \frac{tt}{x}$, & $\overline{PT}^2 = xx - 2tt + \frac{t^4}{xx},$

Ideoque $\overline{MT}^2 = xx - cc - 2tt + \frac{t4}{xx} + \frac{ccxx}{tt}.$

Triangula autem MPT, CET, ſunt ſimilia,

Ergo $\overline{MT}^2 : \overline{MP}^2 :: \overline{CT}^2 : \overline{CE}^2,$

Vel $xx - cc - 2tt + \frac{t^4}{xx} + \frac{ccxx}{tt} : \frac{ccxx}{tt} - cc :: \frac{t^4}{xx} : \frac{-t^4cc}{t^4 - ttxx - ccxx} = \overline{CE}^2.$

* Art. 138. Porro $* \overline{CS}^2 = \overline{CM}^2 + \overline{CB}^2 - \overline{CA}^2 = xx + \frac{ccxx}{tt} - tt;$

Et $\overline{CE}^2 \times \overline{DM}^2$, vel $\overline{CE}^2 \times \overline{CS}^2 = \frac{t^8cc - t^6ccxx - t^4c^4xx}{t^6 - t^4xx - ttccxx} = ttcc,$

Unde $\overline{CE}^2 \times \overline{DM}^2 = \overline{CA}^2 \times \overline{CB}^2$, vel CE × DM $=$ CA × CB.

COROLLARIUM.

146. HINC parallelogramma omnia circa datæ Hyperbolæ diametros quaſvis conjugatas deſcripta, erunt inter ſe æqualia.

LIBER

LIBER QUARTUS
DE TRIBUS
SECTIONIBUS CONICIS.

DEFINITIO.

SINGULA quæque Curva in præcedentibus libris deſcripta (videlicet PARABOLA, ELLIPSIS, HYPERBOLA, vel HYPERBOLÆ OPPOSITÆ) generaliter dici ſolet CONICA SECTIO.

PROPOSITIO I.

THEOREMA.

147. SI *in Ellipſi per terminum alterum* A *diametri cujuſvis* Aa (*vel primæ diametri* Aa, *ſi ſectio fuerit Hyperbola*) *agatur recta* AG *ordinatis parallela, æqualis autem parametro ad diametrum illam pertinenti; & a termino altero* a *ducatur recta* aG, *ordinatam quamvis* PM (*ſi opus, productam*) *in puncto* O *ſecans, dico quadratum Ordinatæ* PM *æquari rectangulo* AP × PO; Fig. 54 & 55.

Scilicet $\overline{PM}^2 = AP \times PO$.

Nam $Aa : AG :: Pa : PO :: AP \times Pa : AP \times PO$,

Sed * $Aa : AG :: AP \times Pa : \overline{PM}^2$,

Unde $\overline{PM}^2 = AP \times PO$.

* Art. 49. & 89.

COROLLARIUM.

148. HINC quadratum Ordinatæ PM ad diametrum A*a* (quod in Parabola (Fig. 56.) ſemper *æquale eſt rectangulo ſub abſciſsâ AP & parametro AG) ſemper minus eſt in Ellipſi, majus vero in Hyperbola, quam prædictum rectangulum ſub abſciſſa & parametro. Et ob has proprietates, Sectionibus Conicis nomina impoſuit Apollonius, ita ut PARABOLA æqualitatem, ELLIPSIS defectum, HYPERBOLA autem exceſſum quadrati Ordinatæ denotaret, cum rectangulo ſub Parametro & abſciſsâ comparati.

* Art. 7. & 23.

PROPOSITIO II.

THEOREMA.

Fig. 57, & 58.

149. IN *Ellipſi omnis diameter* Aa, *& in Hyperbola omnis prima diameter* Aa, *bifariam ſecatur in centro* C, *neque Sectioni in pluribus, quàm duobus punctis occurrit.*

Hoc in articulis 58 libri ſecundi, 104 & 111 tertii demonſtratur.

PROPOSITIO III.

THEOREMA.

150. UNA *tantum tangens* LAL *tranſire poteſt per datum Conicæ Sectionis punctum* A.

Hoc in articulis 24 libri primi, 57 ſecundi, & 116 tertii oſtenditur.

PRO-

PROPOSITIO IV.

THEOREMA.

151. SI *in Sectione Conicâ per punctum quodvis* P *diametri* Aa (*productæ in Hyperbola, quando* Aa *eſt prima diameter*) *agatur recta* MPM *iſtiuſce diametri Ordinatis parallela; dico hanc ſectioni occurrere in duobus punctis* M *&* M, *a puncto* P *æquidiſtantibus;* Fig. 59, 60, 61. 62.

Et vice versâ, ſi recta MM *ad Sectionem terminata bifariam ſecetur a diametro* Aa *in puncto* P *non centro, ea iſtiuſce diametri Ordinatis parallela erit.*

Hoc oſtenſum eſt in articulis 10, 12, 23, libri primi; 51, 53, 63, libri ſecundi; & 91, 93, & 126 libri tertii.

COROLLARIUM.

152. HINC ſi recta MM ad Sectionem Conicam terminata a diametro Aa bifariam ſecetur in puncto P non centro, omnes rectæ ipſi MM parallelæ, & ad Sectionem terminatæ ab eadem diametro biſecabuntur.

PROPOSITIO V.

THEOREMA.

153. SI *duæ rectæ* MM, NN *ſibi mutuo parallelæ, & ad Sectionem terminatæ, bifariam ſecentur in punctis* P *&* Q, *& per ea puncta ducatur recta* Aa; *dico hanc rectam eſſe diametrum.*

Diameter enim per medium punctum P rectæ MM ducta, *tranſibit etiam per medium Q ipſius NN.

* Art. 152.

COROLLARIUM I.

* Def. 7. I. 154. DUCTA aliâ quâvis diametro D*d*, conſtat * Se-
ctionem Conicam eſſe Parabolam, quando D*d* ipſi A*a*
* Def. 9. II. eſt parallela; * Ellipſin, quando D*d* ipſi A*a* intra Sectio-
* Def. 9. III. nem occurrit; Hyperbolam * vero, vel Hyperbolas op-
poſitas, quando D*d* ipſi A*a* extra ſectionem in puncto
C occurrit; ita tamen, ut in his duabus poſtremis caſi-
bus punctum concurſûs C ſit centrum.

Quando tota Ellipſis datur, centrum ejus facile innoteſcet, ductâ diametro A*a*, eâque bifariam ſectâ in puncto C. Eâdem prorſus ratione inveſtigatur centrum, datis Hyperbolis oppoſitis.

COROLLARIUM II.

155. HINC, datâ Conicâ Sectione, ita & puncto quovis O in eodem plano, ſemper duci poterit per illud punctum diameter D*d*; ſi enim Sectio ſit Parabola, agatur per datum punctum recta D*d* diametro cuivis A*a* parallela; ſi Ellipſis, vel Hyperbola, vel Hyperbolæ oppoſitæ, ducatur recta D*d* per punctum datum & per centrum C, ope conſectarii præcedentis inventum.

COROLLARIUM III.

156. HINC recta MM Sectioni Conicæ non niſi in
duobus punctis M & M occurrere poteſt; ductâ enim per
* Art. 151. medium P ipſius MM diametro A*a*, * conſtat rectam MM
iſtiuſce diametri Ordinatis eſſe parallelam, ideoque Se-
ctioni in duobus tantum punctis M & M occurrit.

Si recta per centrum C tranſit, res patet per articulum 149.

Corollarium IV.

157. Data Ellipſi, vel Hyperbola, invenire binas ejuſdem diametros conjugatas A*a*, B*b*; & Aſymptotos CG, C*g*, ducere, quando Sectio fuerit Hyperbola. Fig. 69, 70.

Inventâ, ope parallelarum MN, NN, diametro A*a*, ductâque per centrum C rectâ B*b* ipſis MM, NN, parallelâ, liquet *diametros A*a*, B*b*, eſſe conjugatas, ex eò quod rectæ MM, NN, a diametro A*a* biſariam ſectæ ad diametrum illam parallelæ erunt. * Def. 13. II. 15. III.

Jam verò ad ducendas Aſymptotos CG, C*g*, fiat $\mathrm{AP} \times \mathrm{P}a : \overline{\mathrm{PM}}^{2} :: \overline{\mathrm{CA}}^{2} : \overline{\mathrm{CB}}^{2}$ vel $\overline{\mathrm{C}b}^{2}$; vel (quod eodem recidit) ut media proportionalis inter AP & P*a* ad PM, ita CA ad CB vel C*b*. Ductis igitur rectis AB, A*b*, agantur per centrum C rectæ indefinitæ C*g*, CG, ipſis AB, A*b*, parallelæ, & erunt Aſymptoti quæſitæ; conſtat *enim B*b* magnitudinem eſſe ſecundæ diametri B*b*, ipſi A*a* conjugatæ; unde, per definitiones 14 & 15 libri tertii, liquet propoſitum. * Art. 88. & 126.

PROPOSITIO VI.

Problema.

158. Data *Sectione Conica, ita & ipſius diametro* A*a*, *invenire iſtiuſce diametri Ordinatarum ſitum.* Fig. 59, 60, 61, 62.

Actis ad diametrum datam A*a* parallelis, ſectioni in punctis M & M occurrentibus; dico rectam MM, quæ diametrum datam in puncto P ſecat, Ordinatam eſſe ex utrâque parte iſtiuſce diametri, ſi modo punctum P non fuerit centrum.

Recta enim MM ex hypotheſi biſariam ſecatur a di-
* Art. 151. ametro Aa in puncto P, ideoque *erit Ordinata ex u-
traque parte diametri Aa.

Hâc ratione ſemper licet invenire ſitum Ordinatæ
Fig. 59 & 61. PM ad diametrum Aa; in Parabola enim, & in Hy-
perbola quando diameter data Aa fuerit prima, liquet,
quocunque intervallo rectæ diametro Aa parallelæ a Se-
ctione diſtant, eas Sectioni tandem occurrere in puncto
aliquo M, ex eò quod Sectio magis magiſque in infini-
* Art. 11. 23. 92 & 126. tum *recedat a diametro Aa.

In Ellipſi (Fig. 60) & in Hyperbolis oppoſitis (Fig.
62) quando Aa fuerit ſecunda diameter; liquet ſemper
duci poſſe duas parallelas ex utraque parte diametri Aa,
ſectionem in punctis M & M ſecantes, ita ut recta
MM diametro datæ Aa in puncto P, non centro,
occurrat; ex eo, quòd in Ellipſi Ordinatæ ad diame-
* Art. 52. & 63. trum *Aa eò minores, & è contrariò in Hyperbola eo
* Art. 92. 126. majores *ſunt, quo remotiores a centro C.

COROLLARIUM I.

159. HINC duci poteſt tangens per punctum A Sectio-
* Art. 155. nis Conicæ datum; ducta *enim per illud punctum
diametro Aa, inventâque ad iſtam diametrum Ordinatâ
* Art. 11. 23. 52. 63. 92 & 126. Def. 10, I. 13, II. 7. III. MPM, *liquet, ſi per punctum A agatur ad rectam
MM parallela, eam eſſe tangentem in A.

COROLLARIUM II.

160. DATIS Hyperbolis oppoſitis, vel Ellipſi, ita &
ipſarum diametro quâvis Aa, facile inveniri poteſt dia-

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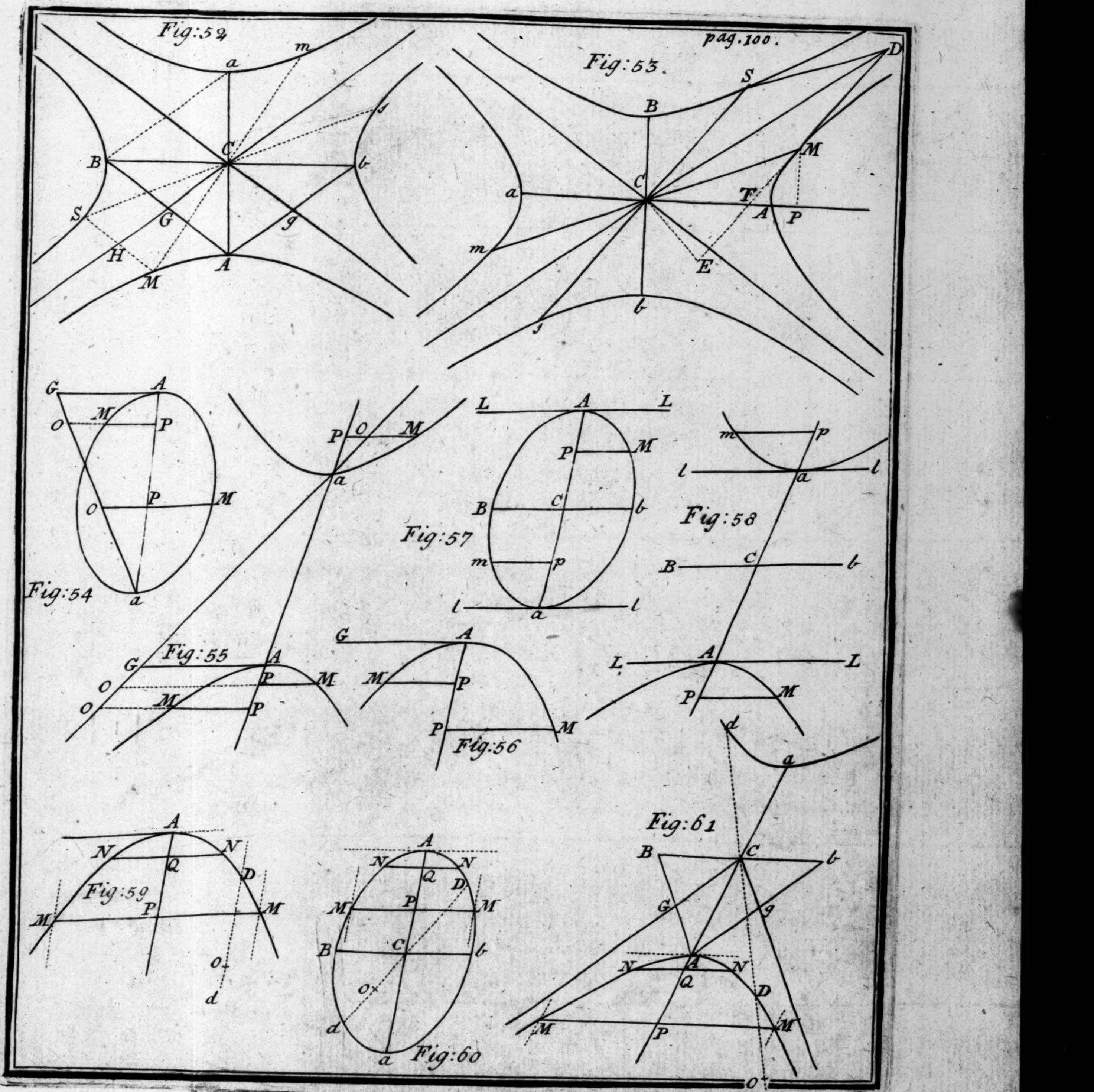
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meter Bb ipſi Aa conjugata. Ducatur ſcilicet per centrum C recta Bb Ordinatis ad illa diametrum parallela.

Sit autem Bb diameter data, cujus conjugata Aa ſit invenienda. Ducta MM ipſi Bb parallela, & ad ſectionem terminatâ, biſecentur rectæ MM, Bb, in punctis P & C, & per P & C agatur Aa, & ea erit diameter quæſita.

Corollarium III.

161. Data Hyperbolâ MAM (Fig. 61.) ita & ſecundæ diametri Bb poſitione, magnitudinem ejus determinare, & Ordinatarum ſitum invenire.

Inveniatur prima diameter Aa, ipſi Bb conjugata per Corollarium præcedens, fiat etiam $AP \times Pa : \overline{PM}^2 :: \overline{CA}^2 : \overline{CB}^2$ vel $\overline{Cb}^2$; & *erit Bb magnitudo ſecundæ diametri Bb, & Ordinatæ ejus diametro Aa parallelæ erunt. * Art. 88. & 126.

Propositio VII.

Problema.

162. Ad *datam Sectionem Conicam, & a puncto* T *extra eam dato, duas tangentes* TM, TM, *ducere.* Fig. 63, 64, & 65.

Pro Parabola.

Ducta per punctum datum T *diametro, quæ Parabolæ in puncto A occurrat, ſumptâque parte AP ipſi AT æquali, agatur *per punctum P recta MM Ordinatis parallela, Parabolæ in punctis M & M *occurrens; junganturque TM, TM, & erunt rectæ TM, TM, *tangentes quæſitæ. PRO-

* Art. 155.
* Art. 158.
* Art. 151.
* Art. 25, & 26.

PRO ELLIPSI.

* Art. 151. DUCTA per punctum datum T * diametro A*a*, ſumtâque
CP tertia proportionali ad CT, CA; agatur per
punctum P recta MM Ordinatis parallela, Ellipſi in duobus
* Art. 151. punctis M & M * occurrens; junganturque TM, TM,
* Art. 65. & * erunt rectæ TM, TM, tangentes quæſitæ.
& 66.

PRO HYPERBOLA ET HYPERBOLIS OPPOSITIS.

DUCTA (Fig. 65.) per punctum datum T diametro
* Art. 161. A*a* (cujus magnitudo * determinanda eſt, ſi fuerit ſecunda
diameter) ſumatur CP tertia proportionalis ad
CT, CA (ex eâdem parte puncti dati T, quoad centrum,
ſi modo punctum illud in angulo ab Aſymptotis
facto contineatur; & ex parte oppoſitâ, quando punctum
T in alterutro angulorum, qui deinceps ſunt, inveniatur)
agaturque per punctum P recta MM Ordinatis
Art. 151. parallela, Hyperbolæ vel Hyperbolis oppoſitis * occurrens
in punctis M & M; junganturque TM, TM; &
* Art. 131. erunt TM, TM, * tangentes quæſitæ.

Si punctum datum ſit centrum, Aſymptoti CG, C*g*,
* Art. 118. erunt * tangentes, quas ducere liceret per articulum 157.
Quòd ſi punctum datum S in altero Aſymptotôn C*g*
fuerit, biſcecetur CS in H, & ducatur recta HM alteri
* Art. 112. Aſymptoto CG parallela, Hyperbolæ in puncto M * occurrens,
* Art. 115. & jungatur SM; & * erit SM una tangentium quæſitarum,
Aſymptotos autem C*g* altera.

COR-

Corollarium.

163. Quoniam recta MPM Ordinatis parallela Sectioni *occurrat in duobus punctis M & M a puncto P æquidiſtantibus, ſequitur duas ſolummodo tangentes duci poſſe a puncto T extra ſectionem dato; unde ſi per punctum concurſûs T duarum tangentium TM, TM, agatur diameter TP, ea bifariam ſecabit rectam MPM. Et vice versâ, ſi diameter TP rectam MPM puncta contactuum jungentem bifariam ſecet, ea per punctum concurſus T tranſibit. * Art. 151.

PROPOSITIO VIII.

Problema.

164. Datis *Conicæ Sectionis Diametro, Parametro ejus, & Ordinatarum ſitu; cognito etiam in Hyperbola, ſi data diameter fuerit prima vel ſecunda; deſcribere modo ſimplici & æquabili tres Conicas Sectiones.*

Pro Parabola.

Sit HAL triangulum iſoſceles, cujus alterum crus AH ſuper diametrum datam AP ex utraque verticis parte productam conſtituatur, alterum vero AL ſuper tangentem indefinitam LAL per punctum A ductam. Jam ſi fingamus baſim HL motu ſibimet parallelo ita ferri, ut una ipſius extremitas L rectam indefinitam LM ipſi AP parallelam ſecum rapiat, altera vero H, rectam FH ipſi AL parallelam, diametri autem AP parametro æqualem; ita tamen, ut ipſius FH extremitas F ſecum ferat rectam FA ipſi HL parallelam, & circa punctum Fig. 66.

fixum

ſixum A mobilem; dico, continuam rectarum FA, LM, interſectionem M (dum recta HL in angulo HAL, & ejus verticali moveatur) Parabolam quæſitam deſignare.

Actâ enim ad Diametrum AP Ordinatâ MP, erunt triangula AHF, APM, ſimilia; ideoque erit, AH vel AL, vel PM : HF :: AP : PM; ergo * $\overline{PM}^{2} = AP \times HF$. unde * erit punctum M ad Parabolam.

* 16. El. 1.
* Art. 7 & 23.

Notandum eſt autem, punctum H ultra diametri AP verticem A cadere debere, quando puncta F, L, ex utrâque parte illius diametri cadunt.

PRO ALIIS SECTIONIBUS.

Fig. 67, 68. EADEM erit conſtructio ac prior, excepto quòd recta LM circa alteram extremitatem *a* datæ diametri A*a* mobilis eſſe debet, quum tamen in Parabola ei parallela ſit. Ponitur datam diametrum in Hyperbola eſſe primam; ſi enim eſſet ſecunda, facile * inveniretur prima ei Conjugata, ita & Parameter.

* Art. 123.

Actâ enim MP ad diametrum A*a* Ordinatâ, triangula *a*PM, *a*AL, & APM, AHF erunt * ſimilia.

* 4. El. 6.

Unde $aP : PM :: aA : AL$ vel AH.

Et $AP : PM :: AH : HF$.

Ideoque $aP \times AP : \overline{PM}^{2} :: aA \times AH : AH \times HF :: aA : HF$,

Ergo $aP \times AP : \overline{PM}^{2} :: aA : HF$, ideoque punctum M * eſt ad Sectionem.

* Art. 49. 63. & 89. 126.

Notandum eſt, puncta H, *a*, ex utraque parte puncti A in Ellipſi cadere debere; & ex eâdem, quando in Hyperbola puncta F, L, ex utrâque parte diametri A*a* cadunt.

PRO-

PROPOSITIO IX.

Theorema.

165. Si *duæ rectæ* MN, AR, *ad Sectionem Conicam* Fig. 69, 70,
terminatæ, & ſibi mutuo occurrentes in puncto aliquo P, 71, 72.
parallelæ fuerint duabus rectis poſitione datis; dico rectangulum MP × PN *eſſe ſemper ad rectangulum* AP × PR *in ratione datâ, quocunque in puncto Sectionis cadant rectæ* MN, AR.

Sint enim in Parabola rectæ contingentes CB, EB, ſibi Fig. 69.
mutuo occurrentes in puncto B, rectis etiam MN, AR, parallelæ, dico eſſe ſemper,

$$MP \times PN : AP \times PR :: \overline{CB}^2 : \overline{EB}^2.$$

Biſectâ enim MN in puncto G, ducatur *diameter CG; * Art. 155.
& per verticem C acta CB ipſi MN parallela, ſectionem in C *continget: ductâ eodem prorſus modo tan- * Art. 11,
gente EB ipſi AR parallela, productâ etiam, uſque dum & 24.
diametro CG in puncto K occurrat; demiſſâque per punctum contactûs E recta EL ad diametrum CG Ordinata, erit *KC = CL; ideoque KB = BE, ob pa- * Art. 25,
rallelas CB, LE. Porro ducatur Ordinata AD, ita & & 26.
recta AF ipſi CG parallela; & ſit KB vel BE $= m$; BC $= n$; CK $= e$; Diametri CG parameter CH $= p$; AP $= x$; PM $= y$; AD $= r$; CD $= s$.

His poſitis, triangula KBC, APF erunt *ſimilia, * Art. 15.
unde,

KB : BC :: AP : PF; Hoc eſt $m : n :: x : \frac{nx}{m} =$ PF.

Eadem ratione erit AF vel DG $= \frac{ex}{m}$.

Unde CG $= \frac{ex}{m} + s$; Et GM vel GN $= y + \frac{nx}{m} + r$.

Et PN $=$ GN $+$ GP $= y + \frac{2nx}{m} + 2r$,

Unde MP × PN $= yy + \frac{2nx}{m}y + 2ry$,

Et $\overline{GM}^2 = yy + \frac{2nx}{m}y + 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr$.

* Art. 8. & 23. Jam verò *CD : CG :: $\overline{AD}^2 : \overline{GM}^2$,

Hoc eſt $s : \frac{ex}{m} + s :: rr : rr + \frac{err}{ms}x = \overline{GM}^2$;

Unde $yy + \frac{2nx}{m}y + 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr = rr + \frac{err}{ms}x$,

* Art. 22. Sed*$\overline{AD}^2 = rr =$ CD × CH $= ps$; ergo $rr + \frac{err}{ms}x = rr + \frac{ep}{m}x$,

Ideoque $yy + \frac{2nx}{m}y + 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr = rr + \frac{ep}{m}x$,

Vel $yy + \frac{2nx}{m}y + 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x - \frac{ep}{m}x = 0$.

Quæ æquatio omnibus Parabolæ punctis æqualiter convenit, modò recta AR ſupra diametrum CG cadat, & punctum concurſus P inter puncta A & R inveniatur.

Jam ſi in hac æquatione ponatur $y = 0$; tum, deletis omnibus terminis, ubi y occurrit, evadet hæc æquatio,

$$\frac{nn}{mm}xx + \frac{2nr}{m}x - \frac{ep}{m}x = 0,$$

Unde dividendo per nnx, erit

$$\frac{x}{mm} + \frac{2r}{mn} - \frac{ep}{mnn} = 0. \text{ Vel } x = \frac{emp}{nn} - \frac{2mr}{n}.$$

* Hyp. Sed, quoniam *y vel PM $= 0$, erit AP vel $x =$ AR.

Ac proinde $AR = \frac{emp}{nn} - \frac{2mr}{n}$;

Sed $PR = AR - AP = \frac{emp}{nn} - \frac{2mr}{n} - x$.

Unde $AP \times PR = \frac{emp}{nn}x - \frac{2mr}{n}x - xx$.

Sed $yy + \frac{2nx}{m}y + 2ry : \frac{emp}{nn}x - \frac{2mr}{n}x - xx :: nn : mm$, nam extremis & mediis in ſe mutuo ductis elicitur æquatio præcedens, unde

$$MP \times PN : AP \times PR :: \overline{CB}^2 : \overline{EB}^2.$$

Et cum tangentes CB, BE, eædem ſemper maneant, quacunque in Sectionis parte cadant rectæ MN, AR, ipſis CB, EB, parallelæ, liquet eſſe ſemper

$$MP \times PN : AP \times PR :: \overline{CB}^2 : \overline{EB}^2.$$

Fieri quidem poteſt, ut pro vario rectarum MN, AR, ſitu varii contingant caſus; ſed cum eorum demonſtratio eadem ſit, exceptis quibuſdam rectis, terminiſque evaneſcentibus, non neceſſe eſt, ut in his fuſius explicandis diutius morer.

PRO ALIIS SECTIONIBUS.

DUCTIS ſemidiametris CO, CB, ipſis MN, AR, parallelis, dico eſſe ſemper Fig. 70 & 71, 72.

$$MP \times PN : AP \times PR :: \overline{CO}^2 : \overline{CB}^2.$$

Ductâ enim *per medium ipſius MN diametro CG, cujus Ordinata ſit MN; demittantur ſuper CG ex punctis A & B, rectæ AD, BE, ipſi MN parallelæ, & agatur AF ipſi CG parallela.

* Art. 155.

Sit $CB = m$; $BE = n$; $CE = e$; $CK = t$; $CO = c$; $AP = x$; $PM = y$; $AD = r$; $CD = s$;

* Art. 15. His poſitis, erit, ob ſimilia * triangula CBE, APF,

Fig. 71, 72. $PF = \frac{nx}{m}$, AF vel $DG = \frac{ex}{m}$. Unde in Hyperbola & Hyperbolis oppoſitis, erit $CG = DG \pm DC = \frac{ex}{m} \pm s$;

GM vel $GN = MP + PF - AD = y + \frac{nx}{m} - r$,

Et $PN = GN + GP = y + \frac{2nx}{m} - 2r$,

Ac proinde $MP \times PN = yy + \frac{2nx}{m}y - 2ry$,

Et $\overline{GM}^2 = yy + \frac{2nx}{m}y - 2ry + \frac{nn}{mm}xx - \frac{2nr}{m}x + rr$.

* Art. 90, & 126. Jam vero * $\overline{CD}^2 \mp \overline{CK}^2 : \overline{CG}^2 \mp \overline{CK}^2 :: \overline{AD}^2 : \overline{GM}^2$,

Hoc eſt, $ss \mp tt : \frac{eexx}{mm} \pm \frac{2esx}{m} + ss \mp tt :: rr : \overline{GM}^2$,

Unde $\overline{GM}^2 = \frac{eerrxx \pm 2emrrsx}{mmss \mp mmtt} + \frac{rrss \mp rrtt}{ss \mp tt}$,

Hoc eſt, $\overline{GM}^2 = \frac{eerrxx \pm 2emrrsx}{mmss \mp mmtt} + rr$;

* Art. 90, & 126. Sed * $\overline{AD}^2 : \overline{CD}^2 \mp \overline{CK}^2 :: \overline{CO}^2 : \overline{CK}^2$ vel $\frac{rr}{ss \mp tt} = \frac{cc}{tt}$; unde ſi, in quantitate $\frac{eerrxx \pm 2emrrsx}{mmss \mp mmtt} + rr$, ſubſtituatur $\frac{cc}{tt}$ loco ipſius $\frac{rr}{ss \mp tt}$, erit $\overline{GM}^2 = \frac{cceexx \pm 2ccemsx}{mmtt} + rr$.

Sed $\overline{GM}^2 = yy + \frac{2nx}{m}y - 2ry + \frac{nn}{mm}xx - \frac{2nr}{m}x + rr$,

Unde $yy + \frac{2nx}{m}y - 2ry + \frac{nntt - ccee}{mmtt}xx - \frac{2nrtt \mp 2cces}{mtt}x = 0$.

Sed

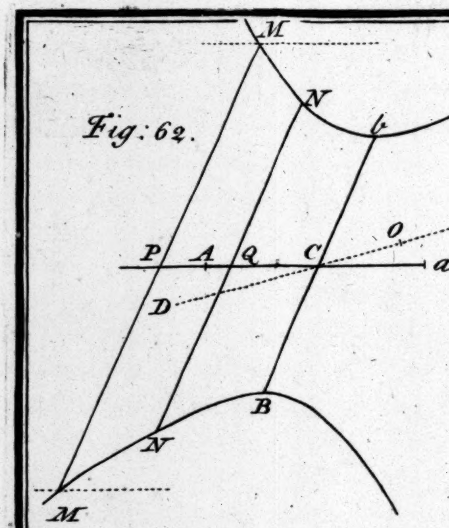


Fig: 62.

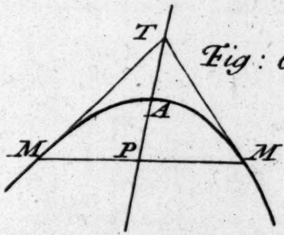


Fig: 63.

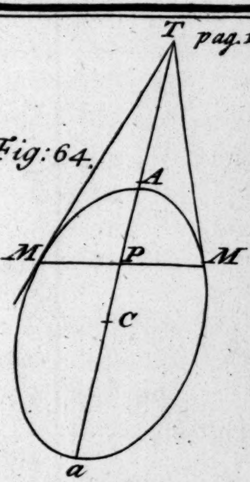


Fig: 64.

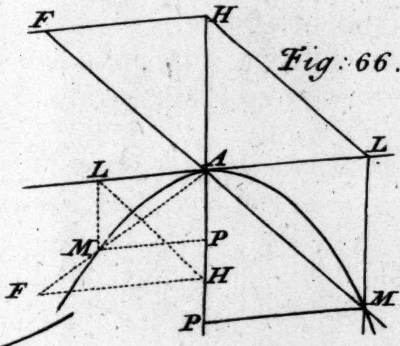


Fig: 66.

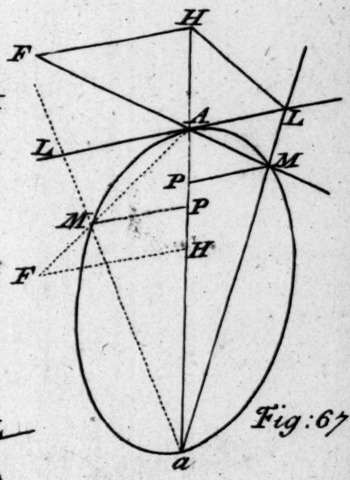


Fig: 67

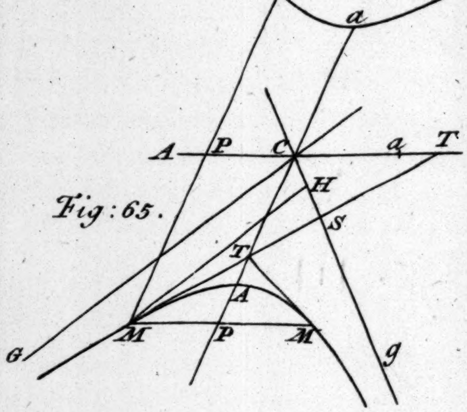


Fig: 65.

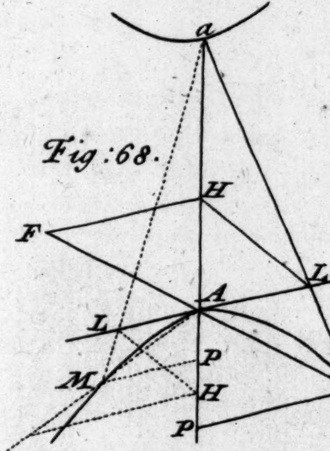


Fig: 68.

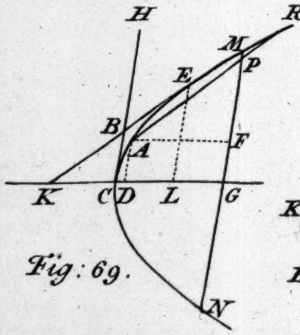


Fig: 69.

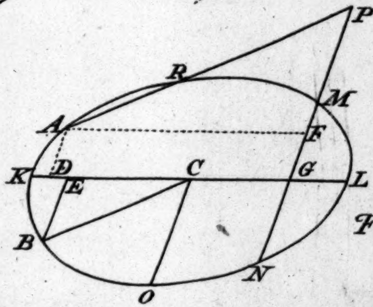


Fig: 70.

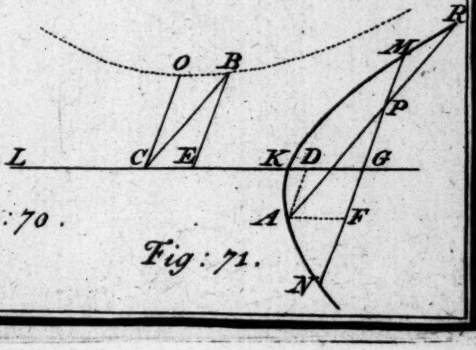


Fig: 71.

Sed * $nntt - ccee = cctt$; (nam $\overline{CE}^2 + \overline{CK}^2 : \overline{EB}^2 :: \overline{CK}^2$ * Art. 88, & 126. $: \overline{CO}^2$, vel $ee + tt : nn :: tt : cc$.) Ergo erit $yy + \frac{2nx}{m}y - 2ry + \frac{cc}{mm}xx - \frac{2nrtt \mp 2cces}{mtt}x = 0$.

Quæ æquatio omnibus Sectionis punctis æquè convenit, modò puncta A & R ex utraque parte diametri CG cadant, punctum autem concursûs P inter puncta A & R inveniatur.

Jam ſi in hâc æquatione ponatur $y = o$, evadet (deletis terminis, ubi y occurrit.)

$$\frac{cc}{mm}xx - \frac{2nrtt \mp 2cces}{mtt}x = o;$$

Unde $x = \frac{2mnrtt \mp 2ccems}{cctt} = \text{AR}$, evaneſcente enim PM vel y erit AP ipſi AR æqualis.

Sed $\text{PR} = \text{AR} - \text{AP} = \frac{2mnrtt \mp 2ccems}{cctt} - x$.

Unde $\text{AP} \times \text{PR} = \frac{2mnrtt \mp 2ccems}{cctt}x - xx$.

Sed * $\frac{2mnrtt \mp 2ccems}{cctt}x - xx : yy + \frac{2nx}{m}y - 2ry :: mm : cc$, * 16. El. 6.

Vel $\text{AP} \times \text{PR} : \text{MP} \times \text{PN} :: \overline{CB}^2 : \overline{CO}^2$, nam extremis & mediis in ſe mutuo ductis, eadem elitur æquatio, ac prior; videlicet $yy + \frac{2nx}{m}y - 2ry + \frac{cc}{mm}xx + \frac{2nrtt \mp 2cces}{mtt}x = o$.

Et cum ſemidiametri CO, CB, eædem maneant, quâcunque in Sectionis parte cadant rectæ MN, AR, ipſis CO, CB parallelæ, liquet eſſe ſemper

$$MP \times PN : AP \times PR :: \overline{CO}^2 : \overline{CB}^2.$$

Demonſtratio pro Ellipſi eadem erit, exceptis quibuſdam rectis.

COROLLARIUM I.

Fig. 73. 166. SI duæ rectæ MN, AR ad Sectionem Conicam terminatæ, ſibi mutuo occurrant in puncto P; ducantur autem rectæ FG, BD, ipſis MN, AR, parallelæ & ad Sectionem terminatæ, ſibi etiam occurrentes in puncto Q; conſtat eſſe ſemper rectangulum MP × PN : AP × PR :: FQ × QG : BQ × QD; ductis enim ſemidiametris CY, CZ, ipſis MN, AR, parallelis, liquet rectas FG, BD iiſdem CY, CZ, eſſe parallelas; unde erit, MP × PN : AP × PR :: $\overline{CY}^2 : \overline{CZ}^2$ :: FQ × QG : BQ × QD.

COROLLARIUM II.

167. SI duæ rectæ AR, BD, ſibi invicem parallelæ & ad Sectionem Conicam terminatæ, alteri rectæ FG ad eandem Sectionem etiam terminatæ in punctis E & Q occurrant, erit FE × EG : AE × ER :: FQ × QG : BQ × QD.

Si enim fingamus rectam MN in corollario præcedenti cadere ſuper rectam FG, liquet rectangulum MP × PN fieri FE × EG; ita & AP × PR fieri AE × ER.

COROLLARIUM III. PRO CIRCULO.

Fig. 74. 168. Ex Theoremate deducitur nota circuli proprietas; nempe ſi intra vel extra circulum per punctum quodvis P ducantur rectæ quotcunque AR, MN, HL &c. ad circumductum circuli terminatæ, omnia rectangula MP × PN, AP × PR, & HP × PL &c. ſibi mutuo æqualia erunt; ductis enim ſemidiametris CB, CO, CD &c. ipſis AR, MN, HL &c. parallelis, conſtat omnia rectangula eſſe ad invicem, ut quadrata ſemidiametrorum CB, CO, CD &c. ex natura circuli æqualium.

Corollarium IV. Pro Parabola.

169. Si per Parabolæ punctum quodvis A ducatur diameter AF, rectæ MN ad Sectionem utcunque terminatæ occurrens in F; dico rectangulum MF × FN æquari rectangulo ſub AF & ſub parametro CH diametri CG, per medium G ipſius MN ductæ. Fig. 75.

Si enim ponamus rectam AP, in Theoremate, ſuper rectam AF cadere, conſtat PF vel $\frac{nx}{m}$ prorſus evaneſcere, ideoque erit $\frac{n}{m} = o$. Deletis igitur omnibus terminis, ubi $\frac{n}{m}$ inveniatur, in æquatione $yy + \frac{2nx}{m}y + 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x - \frac{ep}{m}x = o$, ad Parabolam ſpectante, evadet hæc æquatio, $yy + 2ry - \frac{ep}{m}x = o$; vel $yy + 2ry = \frac{epx}{m}$; ſed eſt * AF $= \frac{ex}{m}$; & CH $= p$; unde AF × CH $= \frac{epx}{m}$. * Art. 165.

Sed eſt etiam MF $= y$; nam evaneſcente PF, eſt MF $=$ MP, & FN $= y + 2r$; unde erit MF × FN $= yy + 2ry$; ideoque MF × FN $=$ AF × CH;

Sed & hoc facilius demonſtrari poteſt;

* Art. 6. Nam *$\overline{GM}^2 = GC \times CH$; & $\overline{AD}^2$ vel $\overline{GF}^2 = DC \times CH$;

Unde $\overline{GM}^2 - \overline{GF}^2 = GC - DC \times CH = AF \times CH$,

* 5. El. 2. Sed *$\overline{GM}^2 - \overline{GF}^2 = MF \times FN$,

Ergo $MF \times FN = AF \times CH$.

Corollarium V. Pro Parabola.

170. Hinc ſi in Parabola per duo puncta quæcunque A & B agantur duæ diametri AF, BP, duabus rectis MN, EL, ſibi mutuo parallelis, & ad Sectionem terminatis occurrentes in punctis F & P; conſtat eſſe $MF \times FN : EP \times PL :: AF : BP$.

Diameter enim CG per medium ipſius MN ducta tranſit *etiam per medium ipſius EL; ac proinde $EP \times PL = BP \times CH$. * Art. 152.

Et $MF \times FN = AF \times CH$;

Sed $AF \times CH : BP \times CH :: AF : BP$,

Ergo $MF \times FN : EP \times PL :: AF : BP$.

2do. Si recta MN ad Parabolam terminata duabus diametris AF, BK in punctis F & K occurrat, erit,

$$MF \times FN : MK \times KN :: AF : BK;$$

Nam $MF \times FN : EP \times PL :: AF : BP$,

Et $EP \times PL : MK \times KN :: BP : BK$,

Ex æquo $MF \times FN : MK \times KN :: AF : BK$.

Unde conſtat *3tio*, ſi duæ rectæ MN, EL ad Parabolam terminatæ & ſibi mutuo parallelæ diametro cuivis BP in punctis K & P occurrant, eſſe,

$$MK \times KN : EP \times PL :: BK : BP.$$

Cor-

COROLLARIUM VI. PRO PARABOLA.

171. HINC per data tria puncta A, M, N, deſcribi poteſt Parabola, cujus diametri AF, CG rectæ poſitione datæ parallelæ fuerint.

Juncta MN, agatur per tertium punctum A, recta AF rectæ poſitione datæ parallela, ipſi etiam MN in puncto F occurrens; & per medium ipſius MN agatur GC ipſi AF parallela; fiat deinde MF × FN ; MG × GN vel $\overline{GM}^2$ :: AF : GC; & ſit CH tertia proportionalis ad rectas* CG, GM; tum *parametro CH, & diametro CG, cujus vertex C, Ordinatæ autem ipſi MN parallelæ fuerint, deſcribatur Parabola; & ea erit quæſita.

* Art. 35. & 36.

Tranſibit *enim per puncta M & N quoniam CH* × CG = $\overline{GM}^2$ vel $\overline{GN}^2$; & per *punctum A, nam MG* × GN : MF × FN :: CG : FA.

* Art. 7 & 23.

* Art. 170.

Porro Diametri AF, CG, rectæ poſitione datæ parallelæ ſunt; & cum Parabola hac ratione deſcripta habeat pro diametro rectam CG, cujus vertex ſit punctum C, & pro parametro determinatam rectam CH, conſtat hanc eſſe ſolam, quæ per data puncta A, M, N, deſcribi poſſit.

COROLLARIUM VII. PRO PARABOLA.

172. SI duæ rectæ AR, MN, ad Parabolam terminatæ ſibi mutuo occurrant in puncto P; fiat autem AP × PR : MP × PN :: $\overline{AP}^2 : \overline{PF}^2$, & jungatur AF; dico hanc rectam eſſe diametrum.

Fig. 69.

Ductis enim contingentibus CB, EB, ad rectas MN, AR parallelis, & per punctum C diametro CG ipſi EB * Art. 165. occurrente in K; erit * $\overline{EB}^2$ vel $\overline{KB}^2 : \overline{BC}^2$ :: AP × PR : MP × PN;

* Hyp. Sed * AP × PR : MP × PN :: $\overline{AP}^2 : \overline{PF}^2$,
Ergo KB : BC :: AP : PF. Unde triangula KBC, APF * 6. El. 6. erunt * ſimilia, ideoque latera AF, KC, ſibi mutuo parallela; ſed CG eſt diameter, ergo AF ipſi CG parallela * Def. 7, I. * erit etiam diameter.

COROLLARIUM VIII. PRO PARABOLA.

173. Ex præcedenti conſectario deducitur methodus deſcribendi Parabolam per quatuor puncta data, A, M, R, N.

Nam ſi quatuor illa puncta duabus rectis AR, MN, ſibi mutuo in P occurrentibus jungantur, & ſi fiat AP × PR : MP × PN :: $\overline{AP}^2 : \overline{PF}^2$; ducatur autem recta AF, * Art. 171. deſcribaturque * per puncta A, M, N, Parabola, cujus diametri rectæ AF parallelæ ſint, liquet hanc eſſe quæſitam; * Art. 165. neceſſe eſt enim recta AP Parabolæ ita * occurrat in R ut ſit AP × PR : MP × PN :: $\overline{EB}^2$ vel $\overline{KB}^2 : \overline{BC}^2$.

* Art. 172. Sed * $\overline{KB}^2 : \overline{BC}^2 :: \overline{AP}^2 : \overline{PF}^2$,

Unde AP × PR : MP × PN :: $\overline{AP}^2 : \overline{PF}^2$.

Fig. 76. Quòd ſi punctum F ex alterâ parte puncti P caperetur; deſcribi poſſet etiam alia Parabola, quæ per quatuor puncta data tranſiret.

Sin vero punctum F ſupra alterutrum punctorum M, N, caderet, una tantum deſcribi poſſet Parabola, quæſitis conditionibus.

Quando autem bina puncta F, F, ſuper bina puncta M, N, cadant, nulla omnino deſcribi poteſt Parabola; ex eo quod in hoc caſu Parabolæ diameter AF per duo ejus puncta tranſiret, quod * fieri nequit. * Art. 11.

Corollarium IX.

Pro Hyperbola vel Hyperbolis Oppositis.

174. Si recta MN ad Hyperbolam vel Hyperbolas oppoſitas terminata, rectæ etiam poſitione datæ parallela, Aſymptoto CB in puncto Q occurrat; ducatur autem per punctum quodvis A recta AP eidem Aſymptoto parallela, ipſi MN in puncto P occurrens; dico rectangulum MP × PN eſſe ſemper ad rectangulum 2 AP × PQ in ratione datâ, quacunque in Sectionis parte cadant rectæ MN, AP. Fig. 77. & 78.

Si enim in Theoremate (Fig. 71, 72) ponamus ſemidiametrum CB fieri Aſymptoton, conſtat latera trianguli CBE tum infinita evadere; ideoque ſi in figuris 77, 78, per terminum K diametri LK per medium ipſius MN ductæ, agatur KS ipſi MN parallela, Aſymptoto CB occurrens in S, conſtitutum erit triangulum CKS, cujus latera ſint finita, quodque triangulo CBE * ſimile erit. * Art. 15.

Unde CK : KS vel *CO :: CE : EB, * Art. 121.
Hoc eſt $t : c :: e : n$, unde * $ce = nt$ ac proinde ſi, loco * 16. El. 6.
ipſius ce, ſubſtituatur valor ejus nt, in æquatione

$$yy + \frac{2nx}{m}y - 2ry + \frac{nntt - ccee}{mmtt}xx - \frac{2nrtt \mp 2cces}{mtt}x = 0,$$

ad Hyperbolam ſpectante, evadet

$$yy + \frac{2nx}{m}y - 2ry - \frac{2nrt \mp 2cns}{mt}x = 0,$$

Vel $yy + \frac{2nx}{m}y - 2ry = \frac{2nrt \pm 2cns}{mt}x$;

Productâ igitur recta AD, ſi opus, uſque dum Aſymptoto CB occurrat in H; erunt triangula CKS, CDH

* 4. El. 6. * ſimilia, unde

CK : KS :: CD : DH, vel $t : c :: s : \frac{cs}{t} = \text{DH}$.

* 34. El. 1. Ideoque AH vel * $PQ = AD \pm DH = r \pm \frac{cs}{t} = \frac{rt \pm cs}{t}$

Unde $2AP \times PQ = \frac{2rt \pm 2cs}{t}x$.

* 16. El. 6. Sed * $yy + \frac{2nx}{m}y - 2ry : \frac{2rt \pm 2cs}{t}x :: n : m$;

Vel MP × PN : 2AP × PQ :: EB : CB, nam extremis & mediis in ſe invicem ductis, evadet æquatio

$$yy + \frac{2nx}{m}y - 2ry = \frac{2nrt \pm 2cns}{mt}x;$$

Et cum rectæ KS, CS, eædem maneant, quacunque in Sectionis parte cadant rectæ MN, AP (ex eo quod diameter LK per medium ipſius MN ducta, tranſit

* Art. 152. * etiam per medium rectarum omnium ipſi MN parallelarum, & ad Sectionem terminatarum) erit ſemper MP × PN : 2AP × PQ :: EB : CB :: KS : CS.

* Fig. 77. Sed & hoc alio modo demonſtrari poteſt.

Sit CK $= t$; KS vel CO $= c$; CS $= m$; CD $= s$; AD vel DI $= r$; AP $= x$; PM $= y$;

* Art. 15. Et erunt triangula CSK, APF * ſimilia,

Unde CS : SK :: AP : PF,

Vel $m : c :: x : \frac{cx}{m} = \text{PF}$; eodem prorſus modo oſtenditur AF vel $\text{DG} = \frac{tx}{m}$,

Ideoque GM vel $\text{GN} = \text{PM} + \text{PF} - \text{GF} = y + \frac{cx}{m} + r$;

Et $\text{CG} = \text{DG} + \text{CD} = \frac{tx}{m} + s$.

Porro ob triangula CKS, CDH, CGQ * ſimilia, erit, CK : KS :: CD : DH; *2. El. 6.

Scilicet $t : c :: s : \frac{cs}{t} = \text{DH}$;

Et CD : DH :: CG : GQ,

$$s : \frac{cs}{t} :: \frac{tx}{m} + s : \frac{cx}{m} + \frac{cs}{t} = \text{GQ};$$

Sed * $\text{MQ} \times \text{QN} = \overline{\text{GQ}}^2 - \overline{\text{GM}}^2$; *6. El. 2.

Unde $\text{MQ} \times \text{QN} = \frac{2ccsx}{mt} + \frac{ccss}{tt} - yy - \frac{2cx}{m}y + 2ry + \frac{2crx}{m} - rr$.

Eſt etiam * $\text{MQ} \times \text{QN} = \text{AH} \times \text{HI} = \overline{\text{DH}}^2 - \overline{\text{DI}}^2$, * Art. 105.

Ergo $\text{MQ} \times \text{QN} = \frac{ccss}{tt} - rr$;

Unde $\frac{2cosx}{mt} + \frac{ccss}{tt} - yy - \frac{2cx}{m}y + 2ry + \frac{2crx}{m} - rr = \frac{ccss}{tt} - rr$,

Vel $yy + \frac{2cx}{m}y - 2ry = \frac{2ccsx}{mt} + \frac{2crx}{m}$,

Ergo * $yy + \frac{2cxy}{m} - 2ry : \frac{2csx}{t} + 2rx :: c : m$, * Art. 43.

Scilicet MP × PN : 2AP × PQ :: KS : CS;

Eadem prorſus eſt demonſtratio pro Hyperbolis oppoſitis, ſignis ſolummodo quibuſvis mutatis.

COROLLARIUM X.

PRO HYPERBOLA ET HYPERBOLIS OPPOSITIS.

175. Ex Corollario præcedenti ſequitur, *1mo*, ſi duæ rectæ MN, HG, ad Hyperbolam vel Hyperbolas oppoſitas terminatæ, Aſymptoto CS in punctis I & Q occurrant; ducantur autem per duo quævis Sectionis puncta A & B rectæ AP, BD, Aſymptoto CS parallelæ, ipſis MN, HG, in punctis P & D occurrentes; liquet eſſe ſemper $MP \times PN : 2AP \times PQ :: HD \times DG : 2BD \times DI$, Ideoque $MP \times PN : AP \times PQ :: HD \times DG : BD \times DI$.

2do. Si duæ rectæ MN, HG, ſibi mutuo parallelæ, & ad Hyperbolam vel Hyperbolas oppoſitas terminatæ Aſymptoto CS in punctis Q & I occurrant; ducatur autem ex Sectionis puncto quovis A recta AO, ipſi CS parallela, ipſis MN, HG, in punctis P & O occurrens; conſtat, ſi ponamus in præcedenti numero rectam BD ſuper AP cadere, eſſe

$MP \times PN : HO \times OG :: AP \times PQ : AO \times OI :: AP : AO$,

Ergo $MP \times PN : HO \times OG :: AP : AO$.

3tio, Si recta HG ad Hyperbolam vel Hyperbolas oppoſitas terminatæ Aſymptoto CS occurrat in I; agantur autem per duo quævis Sectionis puncta A, B, rectæ AO, BD, Aſymptoto CS parallelæ, ipſi HG in punctis O & D occurrentes, erit

$$HO \times OG : HD \times DG :: AO \times OI : BD \times DI.$$

Hoc etiam ſequitur ex numero *1mo*, ſi rectam MN ſuper HG cadere fingamus.

COR-

COROLLARIUM XI.

176. Si ſingamus rectam BD, quæ Sectioni Conicæ in duobus punctis B & D occurrit, motu ſibimet parallelo eo uſque ferri, dum fiat tangens LS; liquet puncta occurſûs B & D, tum in unum coitura ad punctum contactûs L; ideoque punctum contactûs pro duobus interſectionis punctis in ſe mutuo cadentibus haberi poteſt. Fig. 73.

Hoc poſito; conſtat *1mo*, ſi duæ contingentes KS, LS, ſibi mutuo occurrant in puncto S; agantur autem duæ aliæ rectæ MN, AR, contingentibus parallelæ, ad Sectionem terminatæ, & ſibi mutuo occurrentes in puncto P; eſſe $MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2$; hoc enim in Theoremate de Parabola jam oſtenſum eſt. Pro aliis autem Sectionibus, ſi * in primo conſectario ſingamus, rectam FG ſuper tangentem KS cadere, & BD ſuper LS, liquet puncta interſectionis F & G, in puncto contactus K conventura, ita & puncta B, D, in puncto contactûs L; unde rectangulum $FQ \times QG$ evadet quadratum $\overline{KS}^2$, & rectangulum $BQ \times QD$ erit quadratum $\overline{LS}^2$; ac proinde erit $MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2$. * Art. 166.

2do. Si in Ellipſi vel in Hyperbolis oppoſitis ducatur contingens TX ipſi KS parallela rectæ SL in puncto X occurrens, eadem ratione ac in numero præcedenti oſtenditur, eſſe $MP \times PN : AP \times PR :: \overline{TX}^2 : \overline{LX}^2$; cadente enim recta FG ſuper tangentem TX, puncta interſectionis F & Q convenient in T; ita & puncta B & D convenient in L; ideoque $MP \times PN : AP \times PR :: \overline{TX}^2 : \overline{LX}^2$.

Unde ſequitur, ſi tangentes duæ KS, TX, ſibi invicem parallelæ tangenti tertiæ LS occurrant in punctis S, X, erit KS : LS :: TX : LX,

Nam $MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2$,

Et $MP \times PN : AP \times PR :: \overline{TX}^2 : \overline{LX}^2$. Ergo *&c.*

3*tio.* Si in Ellipſi vel Hyperbolis oppoſitis duæ contingentes KS, LS ſibi mutuo occurrant in S; ducantur autem duæ ſemidiametri CY, CZ, contingentibus parallelæ; dico eſſe ſemper KS : LS :: CY : CZ;

Nam $MP \times PN : AP \times PR :: \overline{CY}^2 : \overline{CZ}^2$,

Et $MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2$,

Ergo $\overline{CY}^2 : \overline{CZ}^2 :: \overline{KS}^2 : \overline{LS}^2$, Et CY : CZ :: KS : LS.

4*to.* Si duæ rectæ AR, FG, ad Sectionem Conicam terminatæ, duabus contingentibus KI, LO, occurrant in punctis I & O, dico eſſe ſemper $FO \times OG : \overline{LO}^2 :: \overline{KI}^2 : AI \times IR$, ſi enim fingamus rectam BD in corollario * primo fieri tangentem LO, & rectam MN fieri tangentem KI, res patet.

* Art. 166.

5*to.* Si duæ parallelæ AR, BD, ad Sectionem Conicam terminatæ contingenti KH in punctis I & H occurrant, erit

$\overline{KI}^2 : AI \times IR :: \overline{KH}^2 : BH \times HD$,

Vel $\overline{KI}^2 : \overline{KH}^2 :: AI \times IR : BH \times HD$. Hoc ex ſecundo conſectario deduci poteſt, ſi fingamus rectam FG ſuper tangentem KH cadere.

6*to.* Si in numero præcedenti ponamus Sectionem Conicam fieri Hyperbolam, cujus Aſymptotos ſit tangens HK, liquet rectangula BH × HD & AI × IR fore æqualia, punctum enim contactûs K infinitè *diſtat a punctis H & I; ideoque rectæ infinitæ HK, IK, finitâ quantitate HI inter ſe differentes, pro æqualibus rectis ſunt habendæ.

* Art. 118.

7mo. Si duæ contingentes KS, LS, ſibi mutuo occurrant in puncto S, recta autem AR ad Sectionem terminata, alteri etiam contingentium LS parallela, alteri KS in puncto I occurrat; dico eſſe $\overline{KI}^2 : AI \times IR :: \overline{KS}^2 : \overline{LS}^2$; hoc conſtabit ex corollario *ſecundo, ſi ponamus rectas FG, BD, ſuper tangentes KS, LS, cadere.

* Art. 167.

8vo. Si in Ellipſi vel Hyperbolis oppoſitis duæ contingentes KI, TV, ſibi invicem parallelæ, occurrant in punctis I & V rectæ AR ad Sectionem in A & R terminatæ; dico eſſe $\overline{KI}^2 : AI \times IR :: \overline{TV}^2 : RV \times VA$; hoc etiam ex corollario ſecundo deducitur, ſi parallelæ MN, FG, ſuper tangentes TV, KI cadere ponantur.

9no. Si rectæ HX, PI, ſibi invicem parallelæ & Sectionem ſecantes in punctis X, Y, P, C, tangenti HI occurrant in punctis H & I; erit $\overline{HA}^2$ ad $\overline{AI}^2$ in ratione compoſita ex ratione rectanguli XH × HY ad rectangulum BH × HD, & ex ratione rectanguli BH × HD ad rectangulum PI × IC.

Fig. 80.

Nam *$\overline{HA}^2 : \overline{AI}^2 :: XH \times HY : PI \times IC$.
Sed *XH × HY eſt ad PI × IC in ratione compoſitâ ex ratione ipſius XH × HY ad BH × HD & ex ratione BH × HD ad PI × IC. Ergo &c.

* Num. 5.

* Art. 130.

10mo. Hinc ſi ducantur parallelæ XH, PI, tangenti HI in punctis H & I occurrentes; agatur autem ex puncto H recta HB Sectionem in punctis D & B ſecans, & ipſi PI productæ occurrens in puncto G; erit $XH \times HY : BH \times HD :: CG \times GP : DG \times GB$.

Ductis enim ſemidiametris EA, EL, EZ, ad ipſas PI, HB, HA, parallelis, erit *

* Num. 3. & 7.

$$XH \times HY : \overline{HA}^2 :: \overline{EA}^2 : \overline{EZ}^2,$$

$$\text{Sed } \overline{HA}^2 : BH \times HD :: \overline{EZ}^2 : \overline{EL}^2,$$

$$\text{Ex æquo } XH \times HY : BH \times HD :: \overline{EA}^2 : \overline{EL}^2;$$

* Art. 165. Eſt autem * $CG \times GP : DG \times GB :: \overline{EA}^2 : \overline{EL}^2$,

$$\text{Ergo } XH \times HY : BH \times HD :: CG \times GP : DG \times GB.$$

Fig. 81. 11*mo*. Si rectæ HA, PL, Sectionem contingentes, concurrant in G; agatur autem recta IY tangenti alteri PL parallela, Sectioni in punctis X & Y, tangenti autem alteri in puncto I occurrens, & ducatur utcunque recta IL Sectionem in duobus punctis C & D ſecans, ipſi AP per puncta contactuum A & P ductæ occurrens in S; dico eſſe $XI \times IY : \overline{LP}^2 :: CI \times ID : CL \times LD$.

Ductis enim ſemidiametris EK, EN, EM ad ipſas IY, IL, HA, parallelis, *erit

* Num. 3. & 7.

$$XI \times IY : \overline{IA}^2 :: \overline{EK}^2 : \overline{EM}^2,$$

$$\text{Et } \overline{IA}^2 : CI \times ID :: \overline{EM}^2 : \overline{EN}^2,$$

$$\text{Ex æquo } XI \times IY : CI \times ID : \overline{EK}^2 : \overline{EN}^2;$$

$$\text{Sed } \overline{LP}^2 : CL \times LD :: \overline{EK}^2 : \overline{EN}^2,$$

$$\text{Ergo } XI \times IY : CI \times ID :: \overline{LP}^2 : CL \times LD.$$

Fig. 75. 12*mo*. Sint in Parabola duæ rectæ MN, CH, ſibi invicem parallelæ, quarum altera CH Sectionem in puncto C contingat, altera vero MN ad Parabolam terminetur; ducatur autem per duo quævis Sectionis puncta duæ diametri AF, BO, ipſis MN, CH, in punctis F & O, occurrentes; liquet, ſi in conſectario * quinto, numeris 1*mo*, 2*do*, ponamus rectam EL ſuper tangentem CH cadere, eſſe 1*mo*. MF × FN : $\overline{CO}^2$ :: AF : BO. 2*do*. Productâ FA, uſque dum tangenti CH in puncto Q occurrat; eſſe etiam MF × FN : $\overline{CQ}^2$ :: AF : AQ. * Art. 170.

13*mo*. Sint rectæ MN, KT, ſibi invicem parallelæ; Fig. 79. quarum una KT Hyperbolam in K contingens, Aſymptoto CI occurrat in puncto S; altera vero MN, ad Hyperbolarum oppoſitarum alterutram terminata eidem Aſymptoto in puncto Q occurrat; agantur autem per duo quævis Sectionis puncta A, B, rectæ AP, BT, Aſymptoto CI parallelæ, ipſis MN, KT, in punctis P & T occurrentes; liquet, ſi ponamus in tribus numeris Corollarii decimi, ſecantem GH ſuper tangentem KT cadere, eſſe, 1*mo*. MP × PN : $\overline{KI}^2$ :: AP × PQ : BT × TS. 2*do*. Producta PA, uſque dum ipſi KT in puncto R occurrat, eſſe MP × PN : $\overline{KR}^2$ :: AP : AR. 3*tio*. $\overline{KT}^2$: $\overline{KR}^2$ :: BT × TS : AR × RS.

14*mo*. Si in Hyperbolis oppoſitis duæ contingentes KR, LF, ſibi mutuo parallelæ, Aſymptoto CS in punctis S & V occurrant; agantur autem per duo quævis Sectionis puncta A, B, rectæ AR, BF, Aſymptoto CS parallelæ, contingentibus etiam in punctis R & F occurrentes; dico eſſe, 1*mo*. $\overline{KR}^2$: $\overline{LF}^2$ :: AR × RS : BT × FV. 2*do*. $\overline{KR}^2$: $\overline{LE}^2$ :: AR : AE, quod conſtabit, ſi ponamus ſecantes MN, GH, ſuper rectas contingentes KR, LF, cadere.

PROPOSITIO X.

THEOREMA.

Fig. 82. & 83. 177. SI *in Ellipſi vel in duabus Hyperbolis oppoſitis, quarum centrum* C, *focorum diſtantia* FG, *& axis primus* BD, *ſumatur* CA *tertia proportionalis ad rectas* CF, CB; *ducatur autem per punctum* A *recta* AP *ipſi* BD *normalis, agaturque ex Sectionis puncto quovis* M *recta* MF *ad focum* F, *&* MP *ipſi* AP *perpendicularis*; *dico eſſe ſemper* MP : MF : : BD : FG.

PRO ELLIPSI.

Fig. 82. SIT CB $= a$; CF $= b$; AF $= c$; PM $= y$; AP $= x$; Parameter ipſius KH $=$ KL.

* 47. El. 1. Unde FQ $=$ AF $-$ PM $= c - y$; & * $\overline{MF}^2 = ct - 2cy + yy + xx$.

Sed DF $= a + b$, & FB $= a - b$,

* Art. 42. Unde DF $\times$ FB * $= \overline{CH}^2 = aa - bb$;

* Art. 48. Sed * $\overline{CH}^2 : \overline{CB}^2 :: $ KH : KL,

Ergo KH : KL $:: aa - bb : aa$.

* Hyp. Porro quoniam * CF : CB : : CB : CA, erit etiam CF : CA : : $\overline{CF}^2 : \overline{CB}^2$, vel $\overline{CB}^2 : \overline{CF}^2 :: $ CA : CF,

Unde $\overline{CB}^2 - \overline{CF}^2 : \overline{CB}^2 :: $ CA $-$ CF : CA,

Sive $aa - bb : aa :: c : \frac{aac}{aa-bb} = $ CA,

Ideoque

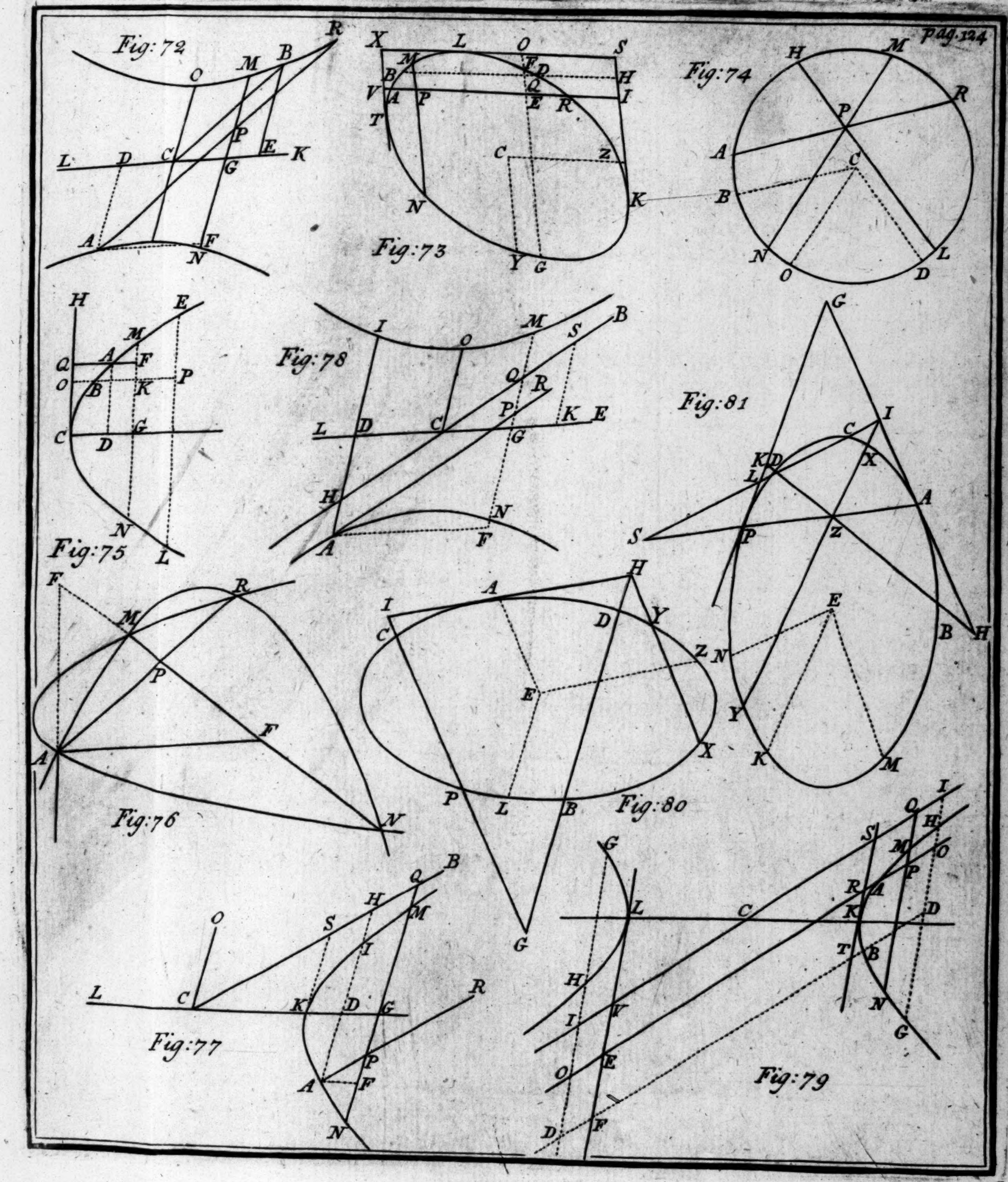


Fig: 72
R O M B P E L D C K G A F N
Fig: 73
X L O S M F D B H Q V A P I E R T C Z K N Y G
Fig: 74
H M P R A C B N O D L
Fig: 75
H E M Q A F O B K P C D G N L
Fig: 78
I O M S B Q R P L D C K E G H N A F
Fig: 81
G C I K D L X A S P Z E B H N Y K M
Fig: 76
F M R P F A N
Fig: 80
H I A C D Y Z E X P L B G
Fig: 77
O Q B H S M I L C K D G R P A F N
Fig: 79
G L C S Q I H M O P R K D T B N G H I V O E D F

Ideoque erit $EM = AC - AQ = \frac{aac}{aa-bb} - y$.

Rurſus quoniam $CF : CB :: CB : CA$, erit etiam * $CB - CF : CF :: CA - CB : CB$,

Id eſt, $BF : FC :: AB : BC$,

Unde $BC : CF :: AB : BF$,

Ideoque * $DF : FC :: AF : FB$; * 18. El. 5.

Ergo * $DF \times FB = \overline{CH}^2 = CF \times FA$. * 16 El. 6.

Et erit $CF : CH :: CH : FA$,

Ac proinde $\overline{CH}^2 : \overline{CF}^2 :: \overline{FA}^2 : \overline{CH}^2$,

Hoc eſt, $aa - bb : bb :: cc : \frac{bbcc}{aa-bb} = \overline{CH}^2$;

Sed * $KE \times EH$ vel $\overline{CH}^2 - \overline{CE}^2 : \overline{EM}^2 :: KH : KL$, * Art. 49.

Hoc eſt, $\frac{bbcc}{aa-bb} - xx : \frac{a^4cc}{\overline{aa-bb}^2} - \frac{2aacy}{aa-bb} + yy :: aa - bb : aa$,

Unde * $\frac{aabbcc}{aa-bb} - aaxx = \frac{a^4cc}{aa-bb} - 2aacy + aayy - bbyy$. * 16. El. 6.

Vel $\frac{a^4cc}{aa-bb} - \frac{aabbcc}{aa-bb} + aaxx + aayy - bbyy - 2aacy = 0$,

Et dividendo æquationem per $aa - bb$, evadet

$\frac{aacc}{aa-bb} - \frac{2aacy}{aa-bb} + \frac{aaxx}{aa-bb} + yy = 0$,

Vel $aacc - 2aacy + aayy + aaxx = bbyy$;

Unde * $yy : cc - 2cy + yy + xx :: aa : bb$, * Art. 43

Viz. $\overline{MP}^2 : \overline{MF}^2 :: \overline{CB}^2 : \overline{CF}^2$,

Sed $\overline{CB}^2 : \overline{CF}^2 :: \overline{DB}^2 : \overline{FG}^2$;

Ergo $MP : MF :: DB : FG$.

Demonſtratio eadem prorſus erit pro duabus Hyperbolis oppoſitis, ſignis quibuſdam mutatis.

COROLLARIUM I.

178. HINC AB : BF :: DB : GF,

* Hyp. Nam * CA : CB :: CB : CF, unde AB : CB :: BF : CF. ideoque AB : BF :: 2CB : 2CF :: DB : GF.

COROLLARIUM II.

Fig. 82. 179. HINC etiam, ſi recta AB major ſit quam BF, Sectio erit Ellipſis, ex eo quod axis primus in hoc caſu major ſit quam diſtantia focorum G & F; ſin autem AB minor ſit quam BF, ac proinde axis primus BD
Fig. 83. minor quam diſtantia focorum F, G, Sectio erit Hyper-
Fig. 84. bola; quod ſi AB æqualis fuerit ipſi BF, Sectio tum
* Art. 1. * erit Parabola.

SCHOLIUM.

180. RECTA indefinita AP vocatur DIRECTRIX non ad Parabolam tantum, ſed & ad Ellipſin vel Hyperbolas oppoſitas.

COROLLARIUM III.

Fig. 85, 86. 181. SI in Sectione Conica duo quævis puncta M & N rectâ MN Directrici in puncto C occurrente jungantur; a foco autem F agantur rectæ FM, FN, FC; dico rectam FC biſecare angulum NFH (complementum ſcilicet anguli NFM ad duos rectos) quando puncta M, N, in Parabola, Ellipſi vel in Hyperbola cadunt; angulum vero NFM, quando puncta illa in Hyperbolis oppoſitis cadunt.

Ductis enim MP, NQ, Directrici CP normalibus, recta autem ND ipſi MF parallela; triangula MPC, NQC & MFC, NDC erunt ſimilia, unde MP : NQ :: MC : NC :: MF : ND, & MP : MF :: NQ : ND;

Sed MP : MF :: NQ : NF; ideoque ND = NF; unde
Fig. 85. in primo caſu angulus NDF, vel ob parallelas CFH, æ-
Fig. 86. quatur angulo CFN; & in ſecundo FDN, vel ei alternus CFM, æquatur angulo CFN.

LIBER

LIBER QUINTUS
DE
SECTIONIBUS CONICIS.
INTER SE MUTUO COMPARATIS.

LEMMA I.

182. QUANTITATES, *ut & quantitatum rationes, quæ ad æqualitatem tempore quovis finito conſtanter tendunt, & ante finem temporis illius propius ad invicem accedunt, quam pro data quavis differentia, fiunt ultimo æquales.*

Si negas; fiant ultimo inæquales, & ſit earum ultima differentia D; ergo nequeunt propius ad æqualitatem accedere, quam pro data differentia D; contra Hypotheſin.

LEMMA II.

183, SIT *Arcus* MN *curvæ cujuſvis* ABG *portio infinitè parva, hoc eſt, minor quavis datâ: jam ſi per terminos ipſius* MN *ad axem vel diametrum* AC *Ordinatim applicentur* MP, NQ; *agantur autem* MR, NS, *ipſi* AC *parallelæ; dico parallelogramma* PQRM *vel* PQNS *haberi poſſe pro ſpatio* PQNM *ſub Ordinatis* PM, NQ, *rectâ* PQ, *& arcu* MN *comprehenſo.* Fig. 87.

Omnia Curvæ cujuſvis puncta ab ejus Diametro vel continuò recedunt, vel ad eam continuo accedunt; vel denique Curva omnis ex pluribus conſtat portiunculis, quarum aliæ a Diametro magis magiſque recedunt, aliæ ad eam magis magiſque accedunt; fieri enim non poteſt, ut Curvæ pars ulla a Diametro æquidiſtet.

His præmiſſis, ſit 1*mo* arcus MN portio Curvæ AMB ab ejus Diametro magis magiſque recedentis; ſumatur ex parte puncti N arcus MO finitæ magnitudinis, ductâque Ordinata OF ipſi MP parallela, agantur rectæ OD, ME Diametro AC parallelæ; & erit ſpatium curvilineum PFOM majus parallelogrammo inſcripto PFEM, circumſcripto autem PFOD minus. Jam ſi punctum O ſuper Curvam verſus M moveri fingarur, liquet parallelogrammum MEOD (differentiam ſcilicet parallelogramorum in arcu OM inſcriptorum & circumſcriptorum) continuo diminui, adeo ut puncto O in M pervento prorſus evaneſcat MEOD; unde ſequitur, ſi diſtantia inter puncta O & N infinitè diminuatur, parallelogrammum MEOD, (jam MRNS factum) minus fore quocunque * Art. 182. dato; ac proinde* parallelogramma PQRM, PQNS, erunt & ſibi invicem & ſpatio curvilineo PQMN æqualia, ideoque pro ſibi invicem uſurpari poſſunt.

2*do*. Sit arcus MN portio Curvæ AMB ad Diametrum ejus magis magiſque accedentis; liquet demonſtrationem eandem eſſe ac in primo caſu, excepto quod parallelogrammum PQNS jam inſcriptum evadat.

3*tio*. Detur Curva ABG ex pluribus conſtans portionibus, quarum aliæ, ſicut AB, a Diametro AG continuò recedant, aliæ vero, ſicut BG, ad eam magis magiſque accedant. Dico fieri non poſſe, ut puncta, ſicut B, has portiones ſeparantia, ſuper arcus MN cadant; ſi enim cadere poſſent, punctum B ad punctum M propius accederet, quam ipſum N; contra Hypotheſin. Liquet ergo hunc caſum in prioribus neceſſario comprehendi. COR-

Corollarium I.

184. Hinc, ſi Ordinatim applicetur recta CB ipſi PM parallela, & ſi fingamus Curvæ portionem AB in arcus infinite parvos dividi, ad inſtar arcus MN; ſpatium ACB ſub rectis AB, AC, & Curvæ portione AB comprehenſum, æquabitur ſummæ parallelogrammorum omnium ipſi PQRM vel PQNS æqualium. Atque ita de omnibus Curvæ ABG partibus dicendum.

Corollarium II.

185. Sit CMDOC figura quæcunque inter parallelas CE, DF, comprehenſa: & ſi inter has parallelas agantur duæ rectæ MO, NL, ipſis CE, DF, parallelæ, quarum diſtantia infinitè parva ſit; dico ſpatium OMNL, in figura CMDOC ſub rectis MO, NL contentum, æquari rectangulo MO × MR vel OS. Fig. 88.

Ductâ enim rectâ AB parallelis CE, DF perpendiculari, ipſis MO, NL etiam parallelis in punctis P & Q occurrente; liquet ſpatium PMNQ æquari rectangulo PMRQ, ita & ſpatium POLQ rectangulo POSQ; unde erit ſpatium OMNL æquale rectangulo OMRS vel OM × PQ.

Corollarium III.

186. Constat ex conſectario præcedenti, ſi duæ quævis figuræ CMDOC, EGFHE inter parallelas CE, DF comprehenſæ ejuſmodi fuerint, ut ducta alicubi rectâ MH ipſis CE, DF, parallela, partes MO, GH, iſtiuſce rectæ in figuris CMDOC, EGFHE, contentæ ſemper ſint in ratione data; conſtat, inquam, binas has

figuras (ſpatia ſcilicet ſub iis comprehenſa) eſſe etiam ad invicem in ratione data; actâ enim recta NK ipſi MH parallela, ita ut diſtantia ejus ab MH infinite parva ſit; ductâ etiam AB ipſis CE, DF, perpendiculari, parallelis autem MH, NK, in punctis P & Q occurrente; * Art. 185. liquet, ſpatium OMNL *æquari rectangulo MO × PQ, ita & ſpatium GHKI æquari rectangulo GH × PQ. Hæc igitur ſpatia erunt ad invicem ut MO ad GH; & cum hoc verum ſit, quacunque in figu-
* 12. El. 6. ræ parte ducatur recta MH, ſequitur *ſummam ſpatiolorum omnium MNLO, hoc eſt, ſpatium CMDOC eſſe ad ſummam ſpatiolorum omnium GHKI, hoc eſt, ad ſpatium EGFHE, in ratione data ipſius MO ad GH.

Eodem ratiocinio probatur, figuræ CMDOC portionem aliquam MDO eſſe ad figuræ EFGHE portionem ſimilem GFH in eadem ratione data: atque ita de partibus reſiduis CMO, EGH dicendum.

Manifeſtum eſt, ſi data ratio fuerit ratio æqualitatis, hoc eſt, ſi partes MO, GH ipſius rectæ MH ſemper æquentur, ſpatia CMDOC, EGFHE, ita & partes MDO, GFH, & CMO, EGH ſemper fore æqualia.

LEMMA III.

Fig. 89. 187. SI *ſuper Curvam quamvis ſumatur arcus* MN *infinitè parvus; & per illius terminos agantur tangentes* MT, NT, *ſibi mutuo occurrentes in puncto* T, *ducaturque ſubtenſa* MN, *& recta* NS *ipſi* MT *productæ normalis; dico ſubtenſam* MN, *vel ſummam tangentium* MT, NT, *vel denique rectam* MS *pro arcu* MN *uſurpari poſſe.*

Neceſſe eſt omnis Curva vel tota concava ſit verſus unam partem, vel ex pluribus portionibus conſtet, quarum aliæ verſus unam aliæ verſus alteram partem Concavæ fuerint; jam vero puncta, quibus hæ portiones ſeparantur, dari *non poſſunt ſuper arcus MN infinitè parvos: alioqui enim propiora eſſent puncto M quàm ipſum N, contra Hypotheſin; poni igitur ſemper poteſt arcum MN eſſe Curvæ portionem verſus unam partem concavam.

* Art. 183. *N.* 3.

Jam ſi ſuper Curvam ex parte puncti N ſumatur arcus MO finitæ magnitudinis, aganturque ſubtenſa OM, tangens OG, & recta OD ipſi NS parallela, erit 1*mo*. Ob triangulum MDO rectangulum ad D, tangens MD minor ſubtenſa MO, & multo minor arcu MNO; adeo ut arcus MNO (ita & ſubtenſa ejus MO) major ſit quàm MD, & minor quam tangentium ſumma MG, OG; 2*do*. Ob arcum MNO verſus eandem partem concavum, liquet, ſi per punctum quodvis N arcus MO ducatur tangens TR, puncta T, R, ubi tangentibus MG, OG, occurrit, erunt inter puncta M, G, & O, G; ideoque angulus OGD triangulo TGR externus, major erit angulo RTG vel NTS.

His poſitis, ſi ducantur rectæ ME, MF, tangentibus OG, NT, parallelæ, ipſi DO in punctis E & F occurrentes; fingaturque punctum O ſuper Curvam verſus M moveri; liquet angulum OGD, vel EMD ipſi OGD æqualem, continuo diminui, adeo ut, puncto O ad M pervento, prorſus evaneſcat; ex eo quod tangens OG cum tangente MG coincidat: unde conſtat rectam ME continuo diminui, donec tandem ipſi MD æqualis evadat; ideoque puncto O in N exiſtente, recta ME tum in MF exiſtens differet a tangente MD quantitate minore quâvis datâ; ac proinde *rectæ TN, TS, ſibi

* Art. 182.

mutuo æquales erunt: unde tangentes MT, TN, ſiſimul ſumptæ, æquales erunt & rectæ MS, & arcui MN, & ſubtenſæ MN.

Corollarium I.

188. Cum angulus FMD, vel NTS ipſi FMD æqualis, infinitè exiguus ſit, poſito punctum N infinite propius ad punctum M accedere, erit in triangulo MTN angulus internus NMT externo NTS minor, infinitè parvus, hoc eſt, minor quovis dato; ideoque nulla recta per punctum M duci poteſt, quæ in angulo TMN cadat; unde patet, rectas MT, NM, ſibi mutuo congruere, ac proinde tangens haberi poteſt pro recta, quæ per duo Curvæ puncta infinitè propinqua tranſeat.

Corollarium II.

189. Si ponamus Curvam quamvis in arcus MN infinitè parvos dividi, liquet ſi loco ipſorum arcuum ſumantur eorum ſubtenſæ, exinde generari Polygonum lateribus numero infinitis, ſingulis etiam infinitè exiguis, quod Polygonum pro ipſa Curva uſurpari poteſt, ab ea enim non *omnino differet.

* Art. 187.

Porro exigua illa Polygoni latera ex utraque parte producta, erunt iſtiuſce Curvæ tangentes, ex eo quod per duo ipſius puncta infinitè propinqua tranſeant.

Lemma IV.

Fig. 90.

190. Si *Circulum* AMG *tetigerit recta* AD, *ſecueritque utcunque recta* APG; *& ſi a quovis puncto* M *in Circuli peripheria ducatur recta* MP *tangenti* AD *parallela, compleaturque*

pleaturque parallelogrammum APMD; *dein accedat punctum* M *ad punctum* A; *dico chordam* AM, *&* *tangentem* AD, *vel* MP, *fore ultimo in ratione æqualitatis.*

Acta enim MG, ob ſimilia triangula AMG, AMD, erit AM : AD :: AG : MG; coeuntibus autem punctis A & M, erit AG ad MG in ratione æqualitatis, adeoque erit etiam AM ad AD in ratione æqualitatis.

COROLLARIUM.

191. Unde cum ſit univerſaliter AP : AM :: AM : AG, coeuntibus A & M, erit AP : MP :: MP : AG, adeoque erit in hoc caſu $\overline{MP}^2 = AP \times AG$.

LEMMA V.

192. SI *figura Elliptica ſecundum longitudines Ordinatarum ad Diametrum quamvis pertinentium dilatetur vel coarctetur, producatur, vel contrahatur, vel ſi ſervatis Ordinatarum longitudinibus & interſectionibus cum Diametro, mutentur utcunque earum inclinationes; dico figuram novam inde genitam nihilominus Ellipticam fore, ſi modo mutationes omnes lineares in longitudinem ubique proportionales fuerint, & angulares æquales.*

Centro C, & diametris quibuſvis conjugatis A*a*, D*d*, utcunque ad invicem inclinatis deſcribatur Ellipſis AMD*a* in qua ſit MP Ordinatim applicata ad diametrum A*a*; dein manentibus ſemidiametri CD & Ordinatarum omnium MP poſitionibus, mutetur longitudo CD in longitudinem CE, & longitudines omnes PM in totidem PN, ita tamen, ut ſingulæ PN ſint ad ſingulas PM ut CE ad Fig. 91.

CE ad CD; dico figuram novam ANE*a* Ellipſin eſſe, ſemidiametris conjugatis AC, CE, deſcriptam;

* Hyp. Nam * PN : PM :: CE : CD,

Unde $\overline{PN}^2 : \overline{PM}^2 :: \overline{CE}^2 : \overline{CD}^2$;

* Art. 48. Eſt autem * $\overline{PM}^2 : AP \times Pa :: \overline{CD}^2 : \overline{CA}^2$,

* 22. El. 5. Ergo * $\overline{PN}^2 : AP \times Pa :: \overline{CE}^2 : \overline{CA}^2$.

Coincidit itaque omni ex parte figura nova ANE*a* cum Ellipſi, ſemidiametris conjugatis AC, CE, deſcriptâ.

Fig. 92. Caſus *2dus.* Servatis jam Ordinatarum longitudinibus & interſectionibus cum diametro, mutetur utcunque angulus ACE in angulum ACF, & anguli omnes APN in totidem APO, ita tamen, ut Ordinatæ omnes PO, CF, parallelæ ſint, & oritur figura nova AOF*a*, quæ etiam Ellipſis erit ſemidiametris AC, CF, deſcripta, ſi modo ſimiles fiant mutationes in altera ſemi-Ellipſi A*da*, nam ob nihil mutatas Ordinatarum longitudines aut interſectiones, erit etiamnum $\overline{PO}^2 : AP \times Pa :: \overline{CF}^2 : \overline{CA}^2$.

COROLLARIUM.

193. FIGURA quævis Elliptica augendo vel minuendo, inclinando vel reclinando diametros ſuas conjugatas in aliam quamcunque ejuſdem generis figuram migrare poteſt.

PROPOSITIO I.

THEOREMA.

194. RATIO *parallelogrammi circa diametros quaſvis conjugatas Ellipſeos cujuſcunque circumſcripti ad figuram Ellipticam inſcriptam eadem ſemper eſt & immutabilis.*

Nam

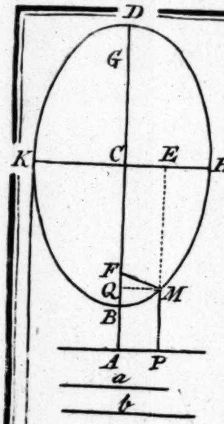


Fig: 82.

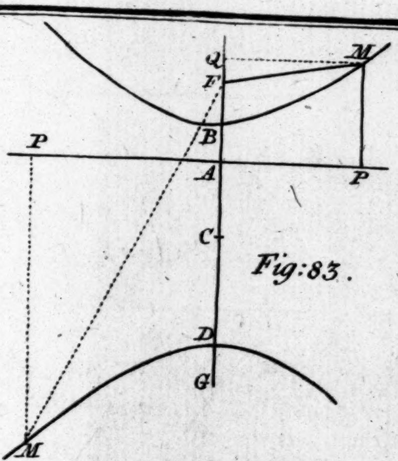


Fig: 83.

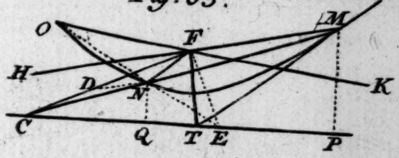


Fig: 85.

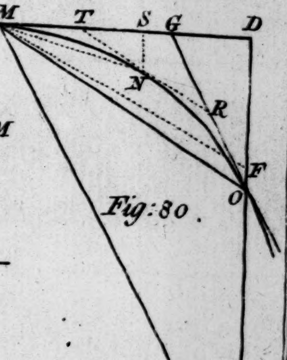


Fig: 80.

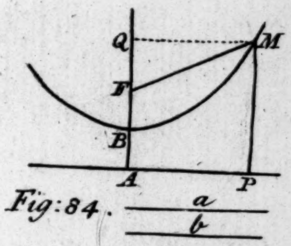


Fig: 84.

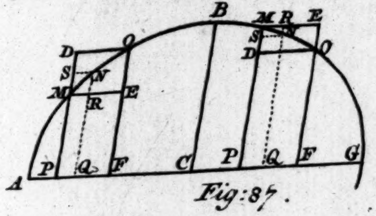


Fig: 87.

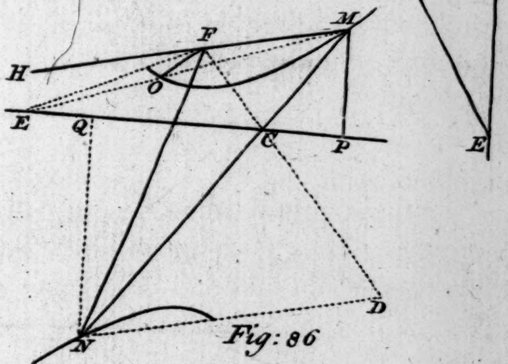
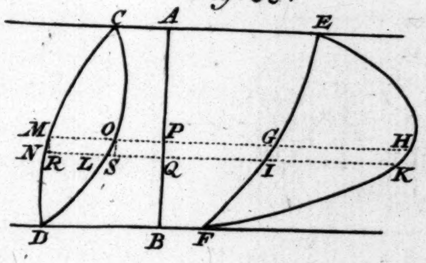


Fig: 86.

Fig: 88.



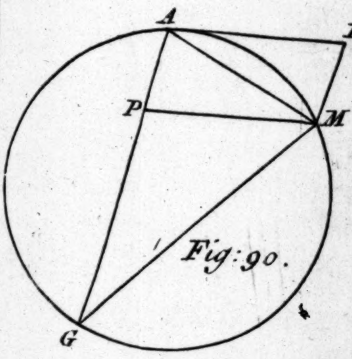


Fig: 90.

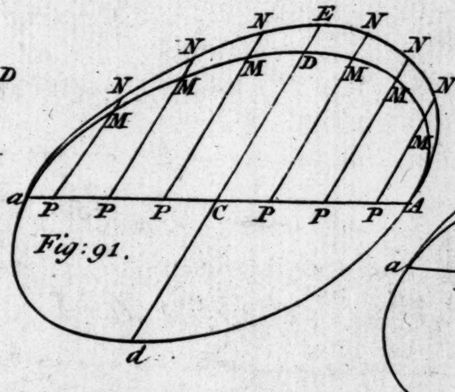


Fig: 91.

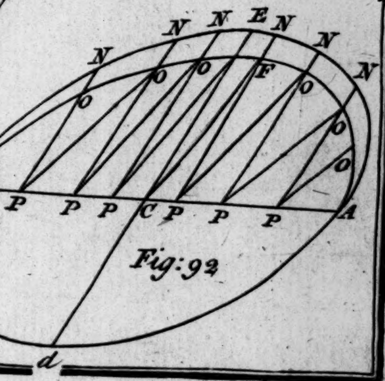


Fig: 92.

Nam ſi dilatando, coarctando, producendo, vel utcunque ordinando figuram inſcriptam, augeatur illa vel minuatur in data aliqua ratione, augebitur vel minuetur in eadem ratione etiam figura circumſcripta; ſed figura quælibet Elliptica augendo vel minuendo, inclinando vel reclinando Diametros ſuas conjugatas in aliam qnamcunque ejuſdem generis figuram *migrare poteſt; patet itaque propoſitum. *Art. 193.

Corollarium I.

195. **Parallelogramma** omnia circa ejuſdem vel æqualium Ellipſium diametros quaſvis conjugatas deſcripta æquantur inter ſe.

Corollarium II.

196. **Ellipseôn** areæ ſunt ad invicem ut parallelogramma circa diametros conjugatas, adeoque ut rectangula ſub axibus.

Corollarium III.

197. **Immutabilis** illa ratio inter Parallelogramma & figuras Ellipticas inſcriptas, eſt ea ipſa, qua quadrati circulo circumſcripti ad inſcriptum circulum, ſeu quam habet 14 ad 11 quam proximè; & ſic reducitur Quadratura Ellipſeos ad circuli quadraturam, nam circulum conſiderare licet tanquam* Ellipſin cujus diametri quævis conjugatæ æquantur inter ſe, & ad rectos ſunt angulos. 6. Archim. Theor.

PRO-

PROPOSITIO II.

THEOREMA.

Fig. 27. 198. Si *per Ellipſeos vel Hyperbolæ* AM *punctum quodvis* A *ducatur diameter* Aa, *cujus parameter* p, *& Ordinata* MP, *& ſi, manentibus diametri & Ordinatarum poſitionibus terminoque* A *& parametro* p, *abire intelligatur terminus alter* a *ad diſtantiam infinitam; dico figuram illam* AM *ultimo transformatam iri in Parabolam Apolloniam cujus diameter eſt* AP, *Parameter ad Diametrum* p, *& Ordinatæ ſingulæ ipſi* PM *parallelæ.*

* Art. 49. Nam ex natura Ellipſeos * erit

$\overline{MP}^2 : AP \times Pa :: p : Aa$,

Sed $p : Aa :: AP \times p : AP \times Aa$,

Unde $\overline{MP}^2 : AP \times p :: AP \times Pa : AP \times Aa :: Pa : Aa$.

Abeat jam punctum *a* ad diſtantiam infinitam, & ratio P*a* ad A*a* fiet ultimo ratio æqualitatis, ergo in hoc caſu
* Art. 6. erit $\overline{MP}^2 = AP \times p$, notiſſima * Proprietas Parabolæ cujus Diameter eſt AP, Parameter ad Diametrum *p*, & Ordinata MP.

SCHOLIUM.

199. Hinc proprietates omnes Parabolæ, ex ſimilibus vel Ellipſeos vel Hyperbolæ proprietatibus derivari poſſunt; quorum exemplum unum aut alterum apponam.

Fig. 27. Ex. 1*mum*. Parabola unico tantum gaudet umbilico F, ob receſſum alterius *f* una cum termino *a* in infinitum,

* Def. 9. II. Ex. 2*dum*. Diametri omnes Ellipſeos * concurrunt in

centro C; ergo Diametri omnes Parabolæ ſunt ſibi invicem parallelæ, utpote non concurrentes, niſi ad diſtantiam infinitam.

3*tium.* Si Ellipſeos vel Hyperbolæ Diametro Aa conjugata ſit Bb, *erit $AF \times Af = \overline{BC}^2$; adeoque $4AF \times Af = \overline{Bb}^2$, ſed ex natura Parametri p, quæ pertinet ad Diametrum Aa, *erit $Aa : Bb : Bb ; p$; adeoque $Aa \times p = \overline{Bb}^2$. Fig. 12, & 13. * Art. 42. * Def. 8. II. & III.

Ergo $4AF \times Af = Aa \times p$.

Et *$p : 4AF :: Af$ ſeu $Fa : Aa$. * 16. El. 6.

Abeat jam terminus *a* ad diſtantiam infinitam, & erit Af ad Aa ultimo in ratione æqualitatis; ergo in Parabola $p = 4AF$, idque ſive AP ſit Axis, ſive alia quævis Diameter.

4*tum.* Producatur Axis Aa ultra terminum A ad E, ita ut æquales fiant AE, AF; centroque *f* & intervallo fE deſcribatur arcus circularis ED, & per M educatur recta fMD, & erit $fD\ (= fE = Aa) = Mf + MF$; ſuducatur utrinque Mf, & erit $MD = MF$; abeat jam punctum *a* una cum puncto *f* ad infinitum, & arcus finitus ED (abeunte centro ad infinitum) jam expandetur in recta axi Aa & rectæ MD perpendiculari; adeoque ſi in axe Parabolæ producto capiatur $AE = AF$, & per E ducatur ED ipſi AE perpendicularis, & ſi ex quovis puncto M ducatur tum MF ad umbilicum F, tum MD axi parallela, rectæ MF, MD ſemper æquales erunt inter ſe. Fig. 93.

5*um.* Tangat recta TMS Ellipſin in puncto M, occurratque ſemidiametro CA productæ in T, & *erit, Fig. 27. * Art. 65.

$CP : CA :: CA : CT$, unde $CP : PA :: CA : AT$,
Et viciſſim, $CP : CA :: PA : AT$.

Abeat centrum C in infinitum, & erit CP ad CA ultimo in ratione æqualitatis; ergo in Parabola erit AP = AT.

Fig. 24. *6tum.* Eſto tangenti MT perpendicularis MG, occurrens Axi A*a* in G; & * erit CP : PG :: A*a* : *p*.
* Art. 69.

Et viciſſim, CP : A*a* :: PG : *p*.

Abeat jam centrum C ad infinitum, & erit CP = $\frac{1}{2}$ A*a*; ergo in Parabola erit PG = $\frac{1}{2}$*p*.

Fig. 27. *7mum.* Per umbilicos F, *f*, ducantur MF, M*f*, & hæ cum tangente æquales angulos TMF, SM*f* efficiunt: capiatur in recta M*f* longitudo quævis finita MO; dein abeat punctum *f* ad infinitum, & recta MO erit ultimo axi A*a* parallela; ergo in Parabola recta MF per umbilicum manentem traducta, & diameter MO æquales cum tangente angulos conficiunt.

PROPOSITIO III.

THEOREMA.

Fig. 94. 200. CIRCULUS, *qui tangit Sectionem Conicam, abſcinditque ex Diametro* Aa *per contactum tranſeunte longitudinem* AG *æqualem Parametro* p *ad Diametrum illam pertinenti, curvaturam habet Sectionis in loco contactus. Et vice verſâ.*

Non enim: ſed, ſi fieri poteſt, coeat alius circulus cujus Diameter AK, cum Sectione Conica AM in loco A, ſecetque Diametrum A*a*, cujus Parameter *p*, in puncto K, quæcunque tandem fuerit longitudo AK, & ad Diametrum A*a* duc Ordinatam MP; & ſi ſectio Ellipſis fuerit vel Hyperbola, * erit,
* Art. 49. & 89.

$$\overline{MP}^2 : AP \times Pa :: p : Aa :: AP \times p : AP \times Aa,$$

Unde $\overline{MP}^2 : AP \times p :: AP \times Pa : AP \times Aa :: Pa : Aa.$

Coeat jam punctum P cum puncto A, & erit Pa ad
Aa ultimo in ratione æqualitatis, ergo in hoc caſu erit
$\overline{MP}^2 = AP \times p$, ut * univerſaliter in Parabola; ſed line- * Art. 7.
ola evaneſcens PM jam communis eſt Ordinata tum Se-
ctioni tum circulo æquicurvo, cujus Diameter AK, pro-
pter ſuppoſitam Curvarum coincidentiam, ergo eſt etiam
$\overline{MP}^2 = AP \times AK = AP \times p$, & $AK = p = {}^{*} AG$; Quod * Hyp.
eſt abſurdum.

SCHOLIUM.

201. Hinc facilè deſcribi poteſt Circulus, qui curvaturam habeat Sectionis in puncto quovis dato; revocatur enim problema ad hoc nempe circulum ducere, qui datam poſitione rectam in dato puncto tangat, & per aliud datum punctum tranſeat.

LEMMA VI.

202. SI *Ellipſin vel Hyperbolam quamvis datam tangat* Fig. 95,
indefinita recta QPT, *&* *ab umbilicis* S *&* F *demittantur ad* 96, 97.
tangentem perpendiculares SQ, FT, *dico rectangulum ſub*
his perpendicularibus æquari quadrato axis ſemiconjugati.

Scilicet $SQ \times FT = \overline{OD}^2$.

Ducantur SP, FP, productaque SP ipſi FT productæ
occurrat in C, & recta TO, quæ biſecat FS in centro
O propter æquales angulos TPF, TPC, biſecabit * etiam * 2. El. 6.
FC in T, & proinde parallela erit ipſi SC, & æqualis
ejus dimidio, vel dimidio ſummæ aut differentiæ rectarum
SP, PF; * vel ſemiaxi principali OA vel OB. Producta ita- * Art. 39.
que TO, donec occurrat ipſi QS (productæ) in t; cir- 80.
culus centro O deſcriptus tranſibit per puncta B, T, t,

* 35. El. 3. A, Q; adeoque * rectangulum tSQ vel FT × SQ erit ubi-
* Art. 42. & 83. que æquale rectangulo dato ASB, *hoc eſt, quadrato $\overline{OD}^2$.

COROLLARIUM I.

203. PERPENDICULUM SQ in data figura erit ut $\sqrt{\frac{SP}{FP}}$. Eſt enim (ob ſim. triang.) SP : SQ :: FP : FT. Ergo $\frac{SQ \times FP}{SP} = FT$, & $\frac{\overline{SQ}^2 \times FP}{SP} = SQ \times FT$; datur itaque quantitas $\frac{\overline{SQ}^2 \times FP}{SP}$, adeoque $\overline{SQ}^2$ erit reciprocè ut $\frac{FP}{SP}$, vel directè ut $\frac{SP}{FP}$; & SQ eſt ut $\sqrt{\frac{SP}{FP}}$.

COROLLARIUM II.

204. CUM ſit SQ × FT $= \overline{OD}^2$, longitudines SQ, OD, FT erunt continuè proportionales, & $\overline{SQ}^2$ erit ad $\overline{OD}^2$ ut SQ ad FT, vel ut SP ad PF; unde erit $\overline{SQ}^2 = \frac{\overline{OD}^2 \times SP}{PF}$.

COROLLARIUM III.

Fig. 98. 205. MANENTIBUS S & A punctis, recedat umbilicus
F ad infinitum, & longitudines SB & PF infinitæ erunt
* Art. 182. cum differentia finita, ergo SB * erit ad PF in ratione
æqualitatis, unde in Parabola eſt $\overline{SQ}^2$ vel $\frac{ASB \times SP}{PF} =$
AS × SP, ideoque SA, SQ, SP erunt continue proportionales.

PRO-

PROPOSITIO IV.

THEOREMA.

206. RECTANGULUM *ſub diſtantiis* SP, PH, *puncti cujuſvis* P *ab utroque Ellipſeos foco, æquatur quadrato ſemidiametri* CD *conjugatæ cum ea quæ per punctum illud ducitur.* Fig. 99.

Scilicet $SP \times PH = \overline{CD}^2$.

Ductâ *enim per punctum P tangente ZPY, demiſſiſque ab umbilicis S, H, & a centro C ad eam perpendicularibus SY, HZ, CR; ſimilia erunt triangula ZPY, HPZ, adeoque erit * Art. 159.

$SP : SY :: HP : HZ :: SP + PH : SY + HZ :: 2AC : 2CR.$

Sed $2AC : 2CR :: AC : CR ::$ *$CD : CB$. * Art. 67. & 145.

Unde $\overline{SP}^2 : \overline{SY}^2 :: \overline{CD}^2 : \overline{CB}^2$.

Eſt autem $\overline{SP}^2 : \overline{SY}^2 :: SP \times PH : SY \times HZ$.

Ergo $SP \times PH : SY \times HZ :: \overline{CD}^2 : \overline{CB}^2$.

Sed *$SY \times HZ = \overline{CB}^2$; ideoque $SP \times PH = \overline{CD}^2$. * Art. 202.

PROPOSITIO V.

THEOREMA.

207. DATO *umbilico Sectionem Conicam deſcribere, quæ datam poſitione rectam* RP *in dato puncto* P *contingat, quæque datam habeat curvaturam in loco contactûs.* Fig. 99.

Eſto Ellipſeos Umbilicus datus S; agatur SP ſecans tum Diametrum DK in E, tum Ordinatim applicatam Q*v* in *x*. Patet EP æqualem eſſe ſemi-axi majori AC,

eo quod acta ab altero Ellipſeos umbilico H linea HI ipſi EC parallela (ob æquales CS, CH) æquentur ES, EI, adeo ut EP ſemi-ſumma ſit ipſarum PS, PI, id eſt
* Art. 72. (ob parallelas HI, PR, & angulos *æquales IPR, HPZ) ipſarum PS, PH, quæ conjunctim axem totum 2AC adæquant.

Ob datum angulum RPS dabitur huic æqualis angulus ZPH, & proinde dabitur poſitione recta PH, quæ per alterum umbilicum H tranſit. In recta PF, vel in eadem (ſi opus) producta, capiatur longitudo PO æqualis Diametro Curvaturæ datæ in P, quâ quidem Diametro deſcribatur circulus PQ ſecans tum Diametrum Sectionis GP in V, tum rectam PS in L, & arcus quam minimus PQ æque pertinebit ad circulum jam deſcriptum ac ad Sectionem jam deſcribendam; porro ex natura circuli, longitudo $PV = \frac{\overline{Qv}^2}{Pv}$ & longitudo $PL = \frac{\overline{Qx}^2}{Px}$ ergo ob æquales $\overline{Qv}^2$, $\overline{Qx}^2$, erit $PV : PL$ (:: $\frac{1}{Pv} : \frac{1}{Px}$ :: $Px : Pv$ :: $PE : PC$ :: $AC : PC$; vel) $SP + PH :$
* Art. 200. $2PC$; ergo $SP + PH \times PL = 2PC \times PV$: oſtenſum *eſt autem longitudinem PV æqualem eſſe Parametro ad Di-
* Art. 206. ametrum $PG = \frac{4\overline{CD}^2}{2PC}$; ergo $2PC \times PV = 4\overline{CD}^2$ *$= 4SP \times PH$; ergo $SP + PH \times PL = 4PS \times PH$. Unde prodit $PH = \frac{SP \times PL}{4SP - PL}$; dantur autem & SP & PL; ergo PH & poſitione & longitudine; & Sectio Conica umbilicis S, H, per punctum P deſcripta tanget rectam RP in

in loco P, & in hoc curvaturam circuli Diametri PO deſcripti habebit. Q. E. I.

Sectio Ellipſis erit, vel Parabola, vel Hyperbola, prout longitudo PL minor fuerit, vel æqualis, vel major quam 4PS.

PROPOSITIO VI.

Theorema.

208. Si *in Sectione Conica ducantur duæ quævis parallelæ* BD, EF, *ad Sectionem terminatæ; junganturque earum termini duabus rectis* BE, DF; *dico ſegmenta* BMEB, DMFD, *ſub Sectionis portionibus, & rectis parallelarum terminos jungentibus comprehenſa, ſibi mutuo æquari.* Fig. 100, & 101.

Productis enim ſubtenſis BE, DF, uſque dum in puncto aliquo G concurrant, ductâque per G & per medium ipſius BD rectâ GH, conſtat rectam EF ipſi BD parallelam ab ipsâ GH bifariam ſecari in puncto K; ita & rectam OO in P; erit igitur recta HK Sectionis *Diameter, cujus Ordinatæ ex utraque parte erunt * Art. 153.
parallelæ BD, EF; ideoque ſi in Sectione per punctum quodvis P ducatur recta MM ipſis BD, EF, parallela, ea Sectioni *occurret in duobus punctis M & M, a pun- * Art. 151.
cto P æquè remotis: unde conſtat partes MO, OM ejuſdem rectæ MM ipſi BD parallelæ, inter ſegmenta BMEB, DMFD contentas, ſibi mutuo æquales eſſe, quacunque in parte inter rectas BD, EF, cadat ipſa MM; ſegmenta *igitur BMEB, DMFD, æquantur. * Art. 186.

Corollarium I.

209. Quoniam MP ipſi PM ſemper ſit æqualis, ſequitur, Fig. 100.

1mo. Trapezia Conica KHBE, KHDF, ſibi invicem æquari.

2do. Quando recta BD Sectionem in puncto A tangit, triangula Conica AKE, AKF æqualia eſſe; ideoque etiam ſegmenta AEMA, AFMA; ex eo quod triangulum AEF in duas partes æquales a recta AK per medium ipſius EF tranſeunte dividatur.

COROLLARIUM II.

Fig. 100. 210. SI, Sectione exiſtente Parabola, Ellipſi, vel Hyperbola, parallelarum BD, EF, termini rectis BF, DE, ſe invicem decuſſantibus jungantur, erunt ſegmenta BFDAB, DEBAD, inter ſe æqualia, triangula enim BFD, BED, inter eaſdem parallelas BD, EF, & ſuper eadem baſi erunt æqualia: unde ſi ex alterâ parte addatur ſegmenmentum DMFD + BADB, & ex alterâ BMEB + BADB, erunt tota BFDAB, & DEBAD ſibi mutuo æqualia; nam ſegmentum DMFD ipſi BMEB æquatur.

PROPOSITIO VII.

THEOREMA.

Fig. 102 103 & 104. 211. SI *in Ellipſi, Hyperbola, vel Hyperbolis oppoſitis, agantur duæ rectæ* BD, EF *ſibi mutuo parallelæ, & ad Sectionem terminatæ; ut & è centro* C *ſemidiametri* CB, CE, CD, CF, *erunt Sectores Elliptici vel Hyperbolici* CBE, CDF, *ſibi mutuo æquales.*

Ducta enim per media H, K, ipſarum BD, EF, diametro CK, triangula CHB, CHD, ita & CKE, CKF, erunt æqualia, utpote quæ communem verticem C, baſes autem HB,

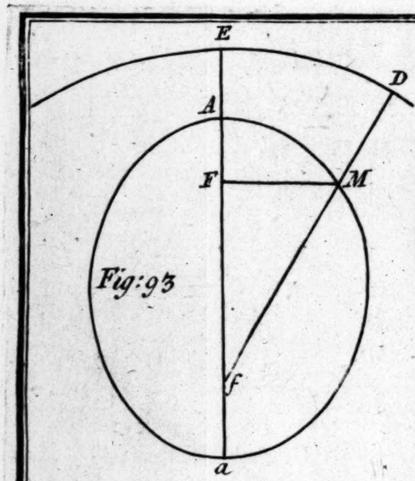
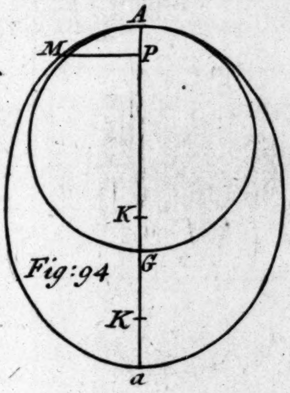


Fig:93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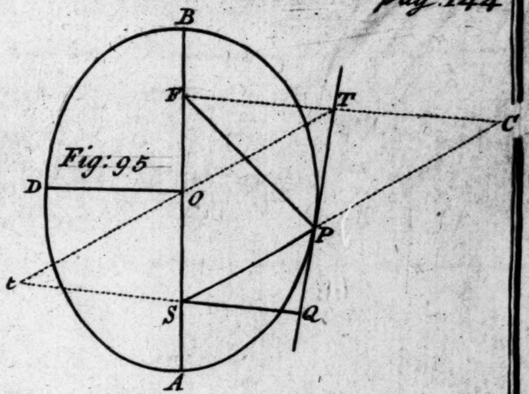
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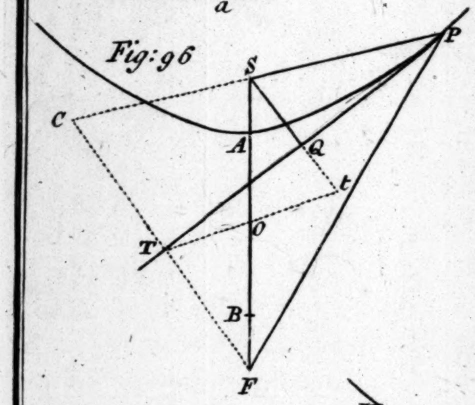
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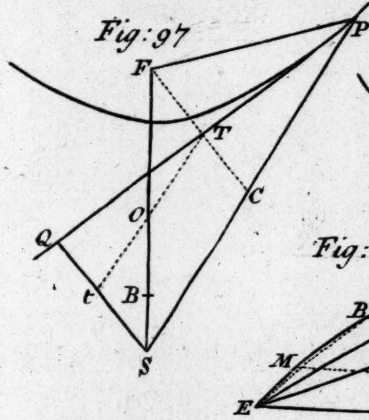
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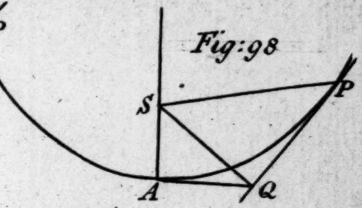
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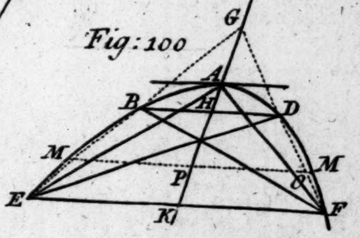
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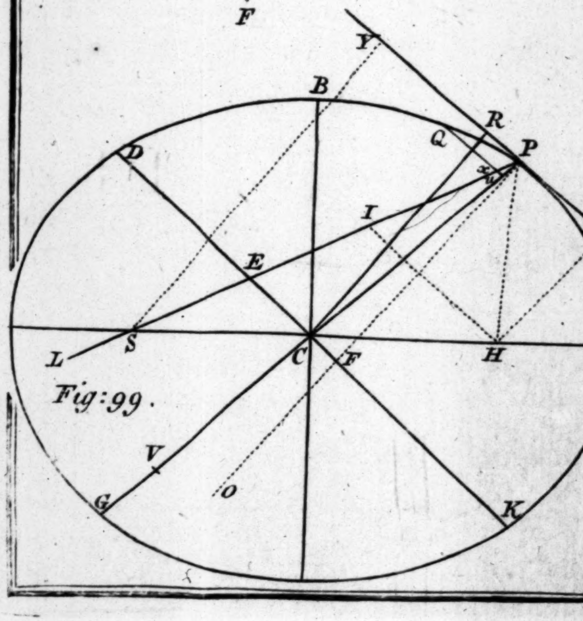
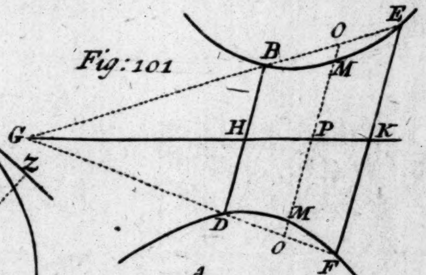


Fig:99.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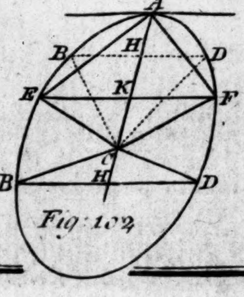
Fig:102

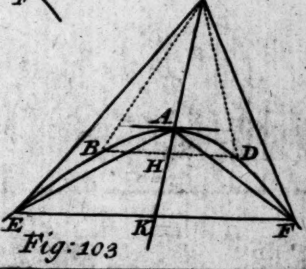
Fig:103

HB, HD; & KE, KF, æquales habeant; ideoque (in Fig. 103.) erit $KHBE + CBE = CKE - CHB = CKF - CHD = KHDF + CDF$, Porro in Fig. 102. 104. $KHBE - CBE = \pm CHB \mp CKE = \pm CHD \mp CKF = KHDF - CDF$, ſed trapezia Conica KHBE, KHDF, *æquantur: unde ſectores Elliptici vel Hyperbolici CBE, * Art. 209. CDF erunt æquales.

COROLLARIUM I.

212. Si in Ellipſi, vel Hyperbola, recta BD ipſi EF Fig. 102. 103. parallela, fieret tangens in puncto A; liquet ſectores CAE, CAF, eſſe inter ſe æquales: productâ enim ſemidiametro CA, uſque dum rectæ EF in puncto K occurrat, illa recta bifariam ſecabitur in illo puncto; ac proinde triangula rectilinea CKE, CKF, æqualia erunt: ſed *triangula * Art. 209. Conica AKE, AKF, æquantur; ergo ſectores CAE, CAF, ſunt inter ſe æquales.

COROLLARIUM II.

213. Hinc facile biſecatur ſector quivis Ellipticus vel Hyperbolicus CEF; ducatur ſcilicet ſemidiameter CA, illius ſectoris ſubtenſam EF in puncto K bifariam ſecans.

PROPOSITIO VIII.

THEOREMA.

214. Si *in ſemicirculo* ADH, *cujus Diameter fuerit Ellipſeos axis major* AH, *agatur per ipſius* AH *punctum quodvis* P *recta* PM *axi normalis, Ellipſi in puncto* M, *circulo autem in puncto* N *occurrens*; *&* *per puncta* M, N, Fig. 105.

ducantur ad centrum C *rectæ* CM, CN; *dico ſectorem Ellipticum* CAM *eſſe ad ſectorem circularem* CAN, *ut* CB *dimidium axis minoris, ad* CA *vel* CD, *dimidium axis majoris.*

Scilicet CAM : CAN :: CB : CA vel CD.

* Art. 50. 63. Nam * $\overline{PM}^2 : \overline{CB}^2 :: AP \times PH : AC \times CH$ vel CA.

* 13. El. 6. Et * $\overline{PN}^2 : \overline{CD}^2 :: AP \times PH : AC \times CH$ vel $\overline{CA}^2$.

Ergo $\overline{PM}^2 : \overline{CB}^2 :: \overline{PN}^2 : \overline{CD}^2$.

Vel PM : PN :: CB : CD. Et cum hoc idem eveniat, quacunque in parte cadat perpendicularis MPN, ſequi-

* Art. 186. tur * ſpatium totum Ellipticum ABHA eſſe ad ſemicirculum ADHA, ita & illius ſpatii partem APM eſſe ad ſemicirculi partem APN, in ratione ipſius CB ad CD vel

* 1. El. 6. CA. Sed triangulum * CPM : CPN :: PM : PN.

Et PM : PN :: CB : CD vel CA. Unde erit APM ± CPM : APN ± CPN :: CB : CD vel CA. (ſigno exiſtente + quando AP minor eſt quàm CA, & — quando AP major eſt quàm CA) ideoque erit ſector Ellipticus CAM ad ſectorem circularem CAN ut CB ad CD vel CA;

COROLLARIUM I.

215. HINC ſector Ellipticus CAM eſt ad ſectorem circularem CAN, ſicut tota Ellipſis ad totum circulum; nam ſemiellipſis eſt ad ſemicirculum in ratione ipſius CB CD vel CA.

COROLLARIUM II.

216. SECTOR Ellipticus CAM æquatur rectangulo ſub arcu AN, & dimidio radii CB; nam area totius cir-

culi * æquatur triangulo cujus baſis eſt peripheria circuli, * 5. Theor. ex Archim. altitudo autem ſemidiameter; unde circuli pars aliqua CAN æqualis eſt triangulo, cujus baſis ſit arcus AN, altitudo autem CD; vel * rectangulo ſub arcu AN & * 41. El. 1. $\frac{1}{2}$CB.

Sed * CAM : CAN :: $\frac{1}{2}$CB : $\frac{1}{2}$CD, * Art. 209.
Et * $\frac{1}{2}$CB : $\frac{1}{2}$CD :: AN $\times$ $\frac{1}{2}$CB : AN $\times$ $\frac{1}{2}$CD. * 1. El. 6.
Ergo CAM : CAN :: AN $\times$ $\frac{1}{2}$CB : AN $\times$ $\frac{1}{2}$CD.
Unde erit CAM $=$ AN $\times$ $\frac{1}{2}$CB.

COROLLARIUM III.

17. HINC ſi in axe majori AH per punctum quodvis G a puncto P diverſum, agatur ad axem normalis GE, Ellipſi in puncto E, circulo autem in puncto F occurrens; erit ſector ACE : ACM :: ACF : ACN.

Nam * ACE : ACF :: CB : CD :: ACM : ACN, * Art. 214.
Ergo ACM : ACE :: ACN : ACF.

Unde facile invenitur ſector Ellipticus ACM, qui ſit ad ſectorem Ellipticum ACE in ratione data. Inveniatur ſcilicet ſector circularis ACN, qui ſit ad ſectorem ACF in ratione data; vel (quod eodem recidit) dividatur arcus ANF vel angulus ACF in ratione data.

PROPOSITIO IX.

THEOREMA.

218. SI *in duabus Hyperbolis* AM, AN, *vel* BM, DN, Fig. 106. & 107. *quarum centrum* C, *ſemidiameter communis recta* AC, *diametri autem conjugatæ* CB, CD, *per punctum quodvis* P *ſemidiametri* CA (*productæ, ſi opus*) *ducatur recta* PM *ipſi* CD *parallela, Hyperbolis occurrens in punctis* M *&* N, *&*

jungantur MC, NC; *dico ſectores Hyperbolicos* CAM, CAN; *vel* CBM, CDN, *eſſe ad invicem in ratione ſemidiametrorum conjugatarum* CB, CD.

* Art. 88. & 126. Nam * $\overline{PM}^2 : \overline{CB}^2 :: \overline{CP}^2 \mp \overline{CA}^2 : \overline{CA}^2$,

Et $\overline{PN}^2 : \overline{CD}^2 :: \overline{CP}^2 \mp \overline{CA}^2 : \overline{CA}^2$,

Unde $\overline{PM}^2 : \overline{CB}^2 :: \overline{PN}^2 : \overline{CD}^2$.

Et PM : PN :: CB : CD.

Et cum hoc idem eveniat, quacunque in parte cadat pa-
* Art. 186. rallela PMN, liquet * ſpatia Hyperbolica APM, APN, vel (in Fig. 107.) CPMB, CPND, eſſe inter ſe, ut CB
* 1. El. 6. ad CD. Sed triangula CPM, CPN, * ſunt ad invicem, ut PM ad PN; vel CB ad CD.
* Fig. 106. Unde * CPM — APM : CPN — APN :: CB : CD.
Hoc eſt, CAM : CAN :: CB : CD.
Et in Fig. 107. erit
CPMB — CPM : CPND — CPN :: CB : CD.
Scilicet CBM : CDN :: CB : CD.

COROLLARIUM.

219. SI duæ ſemidiametri conjugatæ CA, CD, fue-
* Def. 17. Lib. 3. rint æquales, Hyperbola AN vel DN erit * æquilatera; atque ita inventa quadratura ſectorum Hyperbolicorum CAN, vel CDN, haberetur etiam quadratura ſectorum CAM vel CBM, quorum baſes ſunt portiones AM vel alterius Hyperbolæ, poſitâ ſemidiametro conjugatâ CB magnitudinis cujuſcunque; nam ratio ſectorum CAM, CAN, vel CDN, CBM, per rectas CD, CB, deſignata, datur. Unde liquet, datâ Hyperbolæ æquilateræ quadraturâ, dari etiam aliarum omnium Hyperbolarum quadraturam;

draturam; eodem modo, quo datâ circuli quadratura, daretur *etiam omnium quadratura Ellipſium. *Art. 214.

PROPOSITIO X.

Theorema.

220. Si *ſuper Hyperbolæ* EBDF *aſymptoton* CN *ſumantur duæ partes* CK, CL, *quæ ſint ad invicem, ut ejuſdem Aſymptoti duæ aliæ partes* CG, CH; *ducantur autem rectæ* GF, HD, KB, LE, *alteri Aſymptoto* CP *parallelæ, Hyperbolæ etiam in punctis* F, D, B, E, *occurrentes; ut & ſemidiametri* CF, CD, CB, CE; *dico ſectores Hyperbolicos* CBE, CDF, *ſibi invicem æquari.* Fig. 108.

Agantur enim rectæ BD, EF, Aſymptotis in punctis M, O, N, P, occurrentes, & ob parallelas KB, HD, erit

$$MB : MK :: DO : CH.$$

Et ob parallelas LE, GF, erit etiam

$$NE : NL :: FP : CG.$$

Sed *$MB = DO$. Et $NE = FP$; unde erit $MK = CH$; & $NL = CG$. * Art. 103.

Jam vero *$CG : CH :: CK : CL$, *Hyp.

Hoc eſt, $NL : MK :: CK : CL ::$ *$LE : KB$, *Art. 108.

Ergo $LN : LE :: MK : KB$.

Rectæ igitur NE, MB, hoc eſt, EF, BD, erunt parallelæ: ac proinde ſectores Hyperbolici CBE, CDF, erunt *æquales. *Art. 211.

Corollarium. I.

221. Si partes CK, CL, Aſymptoti CN ſint ad invicem in ratione partium CS, CT, Aſymptoti CP; du-

cantur autem KB, LE, Aſymptoto CP, ut & SD, TF, Aſymptoto CN parallelæ, conſtat, ſectores Hyperbolicos CDF, CBE, etiam æquari; actis enim Aſymptoto CP
* Art. 108. parallelis FG, DH; erit *

CG : CH :: HD vel CS : GF vel CT :: CK : CL,

* Art. 220. Ergo CG : CH :: CK : CL. Unde * erunt ſectores CDF, CBE, ſibi mutuo æquales.

COROLLARIUM II.

222. SI ſuper eadem Aſymptoto ſumatur pars CK ipſis CG, CH, tertia proportionalis; eadem ratione, qua in Theoremate, oſtendetur, rectam BF tangenti per pun-
* Art. 212. ctum D ductæ eſſe parallelam; ideoque * erunt ſectores CDF, CDB, inter ſe æquales. Unde ſi ſuper Aſymptoto CL ſumantur partes quotcunque CG, CH, CK, CL, *&c.* in continua progreſſione Geometrica, & ex punctis G, H, K, L, *&c.* agantur rectæ GF, HD, KB, LE, *&c.* alteri Aſymptoto parallelæ, ſectores Hyperbolici CFD, CDB, CBE, *&c.* erunt omnes æquales inter ſe.

Et viciſſim, ſi ſectores Hyperbolici CFD, CDB, CBE, *&c.* fuerint inter ſe æquales, erunt rectæ CG, CH, CL, *&c.* in continua progreſſione Geometrica.

COROLLARIUM III.

223. HINC ſi CH prima ſit duarum mediarum proportionalium inter CG, CL, ſuper Aſymptoton CN; & agantur rectæ GF, HD, LE, alteri Aſymptoto CP parallelæ; ſector CDF erit ad ſectorem CFE ut 1 ad 3. Ita etiam, ſi CH prima ſit trium mediarum proportionalium inter CG, CL, erit ſector CDF ad ſectorem CFE,

ut 1 ad 4. Et univerſaliter, ſi litera *m* numerum quemvis integrum denotet, & ſit GH prima tot mediarum proportionalium inter CG, CL, quot numerus *m* — *i* contineat unitates; erit ſector CDF ad ſectorem CFE ut *i* ad *m*.

SCHOLIUM.

224. Hinc Logarithmorum natura ſatis acuratè explicare poteſt.

Ponamus enim rectam CG unitatem denotare, ipſam autem CL ipſius CG decuplam eſſe, vel 10; dividaturque ſector Hyperbolicus CFE in 10000000000 partes æquales. Jam ſi conſtruatur tabula & in duas columnas dividatur, quarum altera numeros naturales, 1, 2, 3, 4, 5, 6, *&c.* Ordinatim diſpoſitas contineat; altera autem ex adverſo numeros artificiales, ea lege compoſitas, ut ſi CH numerum quemvis naturalem denotet, numerus artificioſus e regione collocatus deſignet, quot partes Sector Hyperbolicus CDF contineat præ numero partium in ſectore CFE contentarum.

Numeri artificiales vocantur Logarithmi numerorum naturalium, quibus reſpondent. His poſitis.

1*mo*. Sint CH, CK, duo numeri naturales in ſe invicem ducendi. Jam ſi in Tabula inveniantur ipſorum CH, CK, Logarithmi, (ſectores CFE, CTB, denotantes) addanque ſibi mutuo; habebitur Logarithmus ſectorem CFE deſignans, cujus è regione collocabitur numerus naturalis CL, factus ex duobus numeris CH, CK, in ſe mutuo ductis.

2*do*. Si numerus CL per numerum CK dividendus ſit, ſubducatur ex Logarithmo CFE dividendi CL, Logarithmus CFB diviſoris CK, & reſiduum erit quoti CH Logarithmus BE vel CFD.

30. Si Radix quæcunque numeri CL fuerit extrahenda, ex gr. Cubica; dividatur in tres æquales partes Logarithmus numeri CL, unde habebitur CFD, Logarithmus numeri CH, radicis quæſitæ.

Omnia hæc conſtant ex eo quod ſectores Hyperbolici
* Art. 220. CFD, CBE, tum ſibi invicem * æquantur, cum eſt CG : CH :: CK : CL; & quod ſectores CFD, CDB, CBE, tum ſibi invicem æquantur, cum eſt CG : CH :: CH : CK :: CK : CL, *&c.*

PROPOSITIO XI.

THEOREMA.

Fig. 109. 225. Si *in utraque Aſymptoto ſumantur partes* CG, CL, *ita &* CR, CS, *ejuſmodi, ut ſint* $\sqrt[m]{CG} : \sqrt[m]{CL} :: \sqrt[n]{CR} : \sqrt[n]{CS}$; *aganturque rectæ* GF, LE, RT, SV, *Aſymptotis parallelæ; dico ſectorem* CFE *eſſe ad ſectorem* CTV, *ut* m *ad* n. *Literæ* m *&* n *numeros quoſvis integros deſignant.*

Fiat enim $\sqrt[m]{CG} : \sqrt[m]{CL} :: CG : CH$.

Ita & $\sqrt[n]{CR} : \sqrt[n]{CS} :: CR : CQ$.

Ducanturque rectæ HD, QN, Aſymptotis parallelæ; &
* Art. 221. erunt ſectores Hyperbolici CFD, CTN, inter ſe * æ-
* Hyp. quales, ex eo quod ſit * CG : CH :: CR : CQ. Jam vero ex natura progreſſionis Geometricæ recta CH erit prima tot mediarum proportionalium inter CG & CL, quot numerus $m - 1$ contineat unitates; ob eandem rationem

erit recta CQ prima tot mediarum proportionalium inter CR & CS quot numerus $n - i$ contineat unitates,
Unde * CFE : CFD :: $m : i$, * Art. 223.
Et CTN vel CFD : CTV :: $i : n$,
Ergo * CFE eſt ad CTV in ratione compoſita ex m * Art. 130.
ad i & ex i ad n, hoc eſt ut m ad n.

COROLLARIUM.

226. HINC dato ſectore Hyperbolico CFE, ita & Hyperbolæ puncto quovis T, facile inveniri poteſt in eadem Hyperbola punctum aliud V ejuſmodi, ut ſit ſector CFE ad ſectorem CTV ut m ad n; ſumatur ſcilicet CS, ita ut ſit $\sqrt[m]{CG} : \sqrt[m]{CL} :: \sqrt[n]{CR} : \sqrt[n]{CS}$, vel quod eodem recidit, $\sqrt[m-n]{CG} : \sqrt[m-n]{CL} :: CR : CS$.

Hoc eſt $CS = CR \times \sqrt[m-n]{\frac{CL}{CG}}_{-n}$.

PROPOSITIO XII.

THEOREMA.

227. SI *per terminos* B, F, *ſectoris cujuſvis Hyperbolici* CBF, *agantur rectæ* BK, FG, *alteri Aſymptoto* CS *parallelæ & ad alteram* CL *terminatæ*; *dico ſectorem Hyperbolicum* CBF *æquari ſpatio Hyperbolico* BKGF, *ſub rectis* BK, FG, *uni Aſymptoto* CS *parallelis*, *ſub alterius Aſymptoti* CL *parte* GK, *& ſub Hyperbolæ denique portione* BF *comprehenſo*. Fig. 110.

Scilicet CBF = BKGF.

Triangula enim CKB, CGF * æquantur; ſublato com- * Art. 107.

muni triangulo CGA (rectæ CB, GF, ſe interſecant in puncto A) & reſidua BKGA, CAF, erunt æqualia; quibus addatur idem ſpatium Hyperbolicum BAF, & erit ſpatium BKGF ſectori CBF æquale.

COROLLARIUM I.

228. DUCTIS rectis BQ, FO, Aſymptoto CL parallelis, & ad Aſymptoton CS terminatis, eodem modo oſtendi poſſet ſectorem Hyperbolicum CBF æqualem eſſe ſpatio Hyperbolico BQOF; unde conſtat, ſpatia vel trapezia Hyperbolica BKGF, BQOF, ſibi invicem æquari.

COROLLARIUM II.

229. UNDE manifeſtum eſt quæcunque in articulis 220, 221, 222, 223, 224, 225, & 226, de ſectoribus Hyperbolicis fuerint demonſtrata, æqualiter obtinere apud hæc trapezia, ſectoribus ipſis æqualia.

PROPOSITIO XIII.

THEOREMA.

Fig. 111. 230. SINT BMF, HND, *duæ Hyperbolæ inter ſe diverſæ, quarum eædem Aſymptoti* CL, CS; *aganturque per duo quævis Aſymptoti unius* CL *puncta* G, K, *rectæ* GEF, KHB, *alteri* CS *parallelæ; dico ſpatium Hyperbolicum* HKGD *eſſe ad ſpatium Hyperbolicum* BKGF, *ſicut dignitas Hyperbolæ* HND *ad dignitatem Hyperbolæ* BMF.

Ducta enim, per punctum quodvis P portionis GK, rectâ PNM ipſis GD, KH parallela, Hyperbolæ BMF occurrente in M, Hyperbolæ autem HND in N; dicatur dignitas Hyperbolæ HND, aa; Hyperbolæ autem BMF, bb; ſit CP $= x$; & * erit PN $= \frac{aa}{x}$, & PM $= \frac{bb}{x}$; unde

* Art. 109.

unde PN : PM :: aa : bb. Et cum hoc verum ſit, quacunque in parte ipſius GK cadat punctum P, liquet *ſpatium Hyperbolicum HKGD : BKGF :: aa : bb. * Art. 186.

Corollarium.

231. Quando dignitates Hyperbolarum HND, BMF, ſunt ad invicem ut numerus m ad numerum n, ſemper inveniri poteſt in Hyperbola HND trapezium Hyperbolicum RSVT æquale trapezio Hyperbolico GKBF in altera Hyperbola BMF, datis rectis CG, CK, CR; liquet *enim trapezium GKHD eſſe ad trapezium GKBF, ut m & n; ideoque in eadem Hyperbola HND inveniendum eſt trapezium RSVT, quod ſit ad trapezium GKHD ut n ad m; id autem fit * capiendo CS ea lege, ut ſit $\sqrt[m-n]{CG} : \sqrt[m-n]{CK} :: CR : CS$. * Art. 225. 228. * Art. 225. 229.

DEFINITIONES.

IV.

Sit AC recta indefinita, cujus terminus fuerit punctum A; ſit AMB curva ejuſmodi, ut, ſi per punctum quodvis P agatur recta MP, quæ cum AC angulum datum APM conſtituat, fiatque indeterminata AP $= x$, PM $= y$, ſit ſemper $ax = yy$ (litera a datam rectam deſignante) conſtat * ex hac Hypotheſi lineam curvam AMB eſſe Parabolam, cujus Diameter fuerit recta AC, Ordinata ad illam Diametrum recta PM, illius autem diametri Parameter a. Fig. 112. * Art. 22.

Jam ſi ponamus curvæ AMB naturam exprimi per æquationem $y^3 = aax$, vel $y^3 = axx$, illa linea curva vocatur Parabola Cubica vel Tertiæ dimensionis; ex eo quòd indeterminatarum x vel y index major ſit tertiæ dimenſionis. Et ſi æquatio fuerit $y^4 = a^3x$, vel, $y^4 = ax^3$; curva AMB vocatur PARABOLA QUARTÆ DIMENSIONIS, quia indeterminata y cujus index omnium major ſit, uſque ad quartam dimenſionem aſcendit. Et ſic in infinitum.

V.

Fig. 113. SIT, ut in definitione præcedenti, recta AC, cujus terminus A; ſit BM curva ejuſmodi, ut, ſi ex ipſius puncto quovis M agatur recta MP, quæ cum AC angulum datum APM conſtituat, fiatque AP$=x$, PM$=y$, ſit ſemper $xy = aa$; (literâ a rectam datam denotante) liquet *hanc curvam fore Hyperbolam, cujus altera Aſymptotos fuerit recta AC, altera vero recta AD ipſi PM parallela, & cujus dignitas vel poteſtas fuerit quadratum aa. Quod ſi æquatio, naturam curvæ BM exprimens, fuerit $xxy = a^3$, illa curva vocatur HYPERBOLA CUBICA, vel TERTIÆ DIMENSIONIS. Et ſi æquatio fuerit $x^3y = a^4$; curva BM erit HYPERBOLA QUARTÆ DIMENSIONIS; ex eo quod factum x^3y quatuor dimenſiones habeat. Et ſic de reliquis omnibus in infinitum.

* Art. 109

COROLLARIUM I.

Fig. 112. 113. 232. SI littera m numerum quemvis integrum deſignet, qui index ſit dignitatis vel poteſtatis ad quam indeterminata AP (vel x) aſcendit; & ſi littera n denotet poteſtatem indeterminatæ PM, vel y, conſtat æquationem $y^n = x^m \times a^{n-m}$ (vel ſimpliciter $y^n = x^m$, factâ $a = 1$) Parabolarum omnium naturam, cujuſcunque tandem dimenſionis fuerint, exprimere. Eodem modo, æquationem $x^my^n = a^{m+n}$ (vel ſimpliciter $x^my^n = 1$, factâ $a = 1$) Hyperbolarum omnium naturam, quæcunque fuerint earum dimenſiones, denotare.

COR-

COROLLARIUM II.

233. Si per ipſius AC terminum fixum A ducatur recta indefinita AD ipſi PM parallela; agaturque MK ipſi AC parallela, ipſi autem AD in puncto K occurrens; fiat $AK = x$; $KM = y$; liquet indeterminatam x, quæ rectam AP vel MK prius deſignarat, jam fieri y; & è contrario indeterminatam y, quæ PM vel AK prius deſignarat, jam fieri x. Unde patet,

1*mo*. Si curva AMB fuerit Apolloniana Parabola, eam exprimi per æquationem $yy = ax$ vel $xx = ay$, reſpectu habito ad rectam AC vel AD; Parabolam etiam cubicam exprimi per æquationem $y^3 = axx$, vel per $x^3 = a^2y$, reſpectu habito ad rectam AC vel AD. Et univerſaliter, ſi curva AMB per æquationem $y^n = x^m a^{n-m}$ exprimatur reſpectu habito ad rectam AC, eandem curvam denotari per æquationem $x^n = y^m a^{n-m}$ (ponitur enim n major eſſe quam m) reſpectu habito ad rectam AD. Fig. 112.

2*do*. Hyperbolam communem vel Apollonianam ſemper exprimi per æquationem $xy = aa$, reſpectu habito ad rectam AC vel AD; Hyperbolam vero cubicam, cujus æquatio fuerit $xxy = a^3$ reſpectu habito ad rectam AC, habere æquationem $xyy = a^3$, reſpectu habito ad rectam AD. Et univerſaliter, Hyperbolam, cujus æquatio reſpectu habito ad AC fuerit $x^m y^n = a^{m+n}$, exprimi per æquationem $x^n y^m = a^{m+n}$ reſpectu habito ad rectam AD. Fig. 113.

COROLLARIUM III.

234. Hinc conſtat, duas eſſe Parabolas Cubicas, quarum una per æquationem $y^3 = aax$ vel $x^3 = aay$ exprimatur; altera vero per æquationem $y^3 = axx$ vel $x^3 = ayy$;

ayy; unicam vero Hyperbolam cubicam $xxy = a^3$ vel $xyy = a^3$. Indeterminatæ enim x & y ſolummodo quatuor primis modis componi poſſunt ad exprimendas Parabolas cubicas vel tertiæ dimenſionis; & duobus tantum modis, ad exprimendas Hyperbolas Cubicas. Jam vero, cum quatuor primæ æquationes ad duas curvas inter ſe diverſas pertineant; duæ autem poſteriores ad unam ſolummodo ſpectent; ſequitur duas eſſe Parabolas cubicas, & unam Hyperbolam cubicam. Eadem ratione inveniri poteſt numerus Parabolarum vel Hyperbolarum quartæ vel quintæ dimenſionis, *&c.*

COROLLARIUM IV.

Fig. 113. 235. RECTÆ indefinitæ AC, AD, non ſunt tantum Aſymptoti Hyperbolæ communis, ſed & omnium Hyperbolarum cujuſcunque dimenſionis. Si enim æquatio $x^m y^n = a^{m+n}$ vel $y^n = \frac{a^{m+n}}{x^m}$ (Ponitur AP $= x$, PM $= y$) naturam cujuſvis Hyperbolæ exprimat, quando puncta ejus quoad rectam AC ſpectentur; liquet, quo major fuerit AP vel x, eo magis y^n, ac proinde PM vel y diminui; adeo ut, ſi x infinitè augeatur, PM vel y prorſus evaneſcat; hoc eſt, Hyperbola BM & recta AC in infinitum productæ magis magiſque ad invicem accedunt, nunquam tamen ſibi mutuo occurrunt, niſi in ipſo, ut ita dicam, infinitatis termino; & hæc * eſt Aſymptoti proprietas.

* Art. 110.

Jam ſi ejuſdem Hyperbolæ puncta quoad rectam AD ſpectentur; habebimus æquationem $x^n y^m = a^{m+n}$ vel $y^m = \frac{a^{m+n}}{x^n}$ (AK $= x$, KM $= y$,) Unde ſequitur, quo major

jor ſit AK vel x, eo minorem fieri KM vel y, & ſic in infinitum; ideoque recta AD eſt etiam Aſymptotos ejuſdem Hyperbolæ.

PROPOSITIO XIV.

Problema.

236. Detur *punctum* M *ſuper ſecundam Parabolam cubicam* AMB, *cujus natura per æquationem* $y^3 = axx$ *exprimatur*; *propoſitum ſit tangentem* MT *ducere*. Fig. 112.

Sit arcus MN infinite parvus, ducaturque NQ ipſi PM, MR ipſi AC parallela: & triangulum MRN ſimile erit triangulo TPM, ex eo quod arcus exiguus MN pro tangentis productæ parte * haberi poſſit. * Art. 189.

Hoc poſito, fiat ſubtangens quæſita TP $= s$; PQ vel MR $= e$; PM $= y$; & erit * RN $= \frac{ey}{s}$. * 4. El. 6.

Unde QN $=$ QR $+$ RN $= y + \frac{ey}{s}$.

Jam ſi loco ipſius y^3 in æquatione $y^3 = axx$, naturam curvæ AMB exprimente, ſubſtituatur cubus ipſius QN vel $y + \frac{ey}{s}$; & loco ipſius xx, quadratum ipſius AQ vel $x + e$, æquatio evadet

$$y^3 + \frac{3ey^3}{s} + \frac{3eey^3}{ss} + \frac{e^3y^3}{s^3} = axx + 2eax + eea,$$ quæ æquatio rationem ipſius AQ ad QN perfectè exprimet.

Quod ſi membra prioris æquationis reſpectivè ſubducantur ex membris prioris, reſiduumque per e dividatur, erit

$\frac{3y^3}{s} + \frac{3ey^3}{ss} + \frac{eey^3}{s^3} = 2ax + ea$. Deletis omnibus terminis, ubi e ocurrit, (exiſtente enim PQ vel e infinitè parvâ, termini illi evaneſcunt) & erit tandem $\frac{3y^3}{s} = 2ax$, unde erit $PT = s = \frac{3y^3}{2ax} = \frac{3}{2}x$, ſubſtituendo, loco ipſius y^3, valorem ejus axx.

SCHOLIUM.

237. Liquet ex præcedenti calculo, ſi loco poteſtatis ipſius y ſubſtituatur eadem potentia ipſius $y + \frac{ey}{s}$, duos primos iſtiuſce poteſtatis terminos ſolummodo requiri; reliqui enim in æquatione ultimò inventa ducti in poteſtates ipſius e, vel ipſam e continent, vel poteſtates ipſius e, ideoque prorſus ſunt delendi. Idem fit, quando loco ipſius x ſubſtituatur eadem poteſtas ipſius $x + e$. Quod ſi omnes poteſtates radicis binomicæ $x + e$ conſtituas, erunt duo primi termini ſecundæ dimenſionis $x^2 + 2ex$; tertiæ, $x^3 + 3exx$; quartæ, $x^4 + 4ex^3$; quintæ, $x^5 + 5ex^4$, & ſic in infinitum; adeo ut duo primi termini poteſtatis cujuſvis m radicis binomicæ $x + e$, ſint $x^m + mex^{m-1}$. Eadem ratione conſtat duos primos terminos poteſtatis cujuſvis n radicis binomicæ $y + \frac{ey}{s}$, eſſe $y^n + \frac{ney^n}{s}$.

Cor-

COROLLARIUM.

238. UNDE, ope æquationis generalis $y^n = x^m a^{n-m}$, vel (facta $a = 1$) $y^n = x^m$, generaliter exprimi poteſt ſubtangens PT (s) Parabolarum omnium cujuſcunque dimenſionis; hoc modo,

Ponantur, in æquatione generali $y^n = x^m$, loco ipſius y^n, duo primi termini poteſtatis n radicis $y + \frac{ey}{s}$, hoc eſt $y^n + \frac{ney^n}{s}$; & loco ipſius x^m ponantur duo primi termini poteſtatis m radicis $x + e$; hoc eſt, $x^m + mex^{m-1}$; unde erit $y^n + \frac{ney^n}{s} = x^m + mex^{m-1}$. Subducantur membra prioris æquationis ($y^n = x^n$) ex membris poſterioris reſiduumque per e dividatur, & erit $\frac{ny^n}{s} = mx^{m-1}$; unde $s = \frac{ny^n}{mx^{m-1}} = \frac{nx^m}{mx^{m-1}} = \frac{n}{m}x$, nam $y^n = x^m$.

PROPOSITIO XV.

PROBLEMA.

239. DUCERE *tangentes ad Hyperbolas cujuſcunque dimenſionis.* Fig. 113.

Manente Propoſitionis præcedentis conſtructione, ponantur, in æquatione generali $x^m y^n = a^{m+n}$ rationem ipſius AP (x) ad PM (y) denotante, loco ipſius x^m, duo primi termini poteſtatis m radicis AQ, ($x + e$) hoc eſt, $x^m + mex^{m-1}$; & loco ipſius y^n, duo primi termini poteſtatis n radicis QN $(y - \frac{ey}{s})$ hoc eſt, $y^n - \frac{ney^n}{s}$, unde oritur hæc æquatio

$x^m y^n + mey^n x^{m-1} - \frac{ney^n x^m}{s} - \frac{mneey^n x^{m-1}}{s} = a^{m-n}$ rationem ipſius AQ ad QN exprimens. Membris igitur prioris æquationis $x^m y^n = a^{m+n}$ ex membris poſterioris reſpectivè ſubductis, dividendo per ey^n, erit $mx^{m-1} - \frac{nx^m}{s} - \frac{mnex^{m-1}}{s} = o$; deleto autem evaneſcente termino $- \frac{mnex^{m-1}}{s}$ ex eo quod rectam PQ (e) infinite parvam in ſe contineat, & erit ordinatis terminis PT, vel $s = \frac{nx^m}{mx^{m-1}} = \frac{n}{m}x$.

Corollarium.

Fig. 112. 113. 240. Hinc in Parabola vel Hyperbola cujuſvis dimenſionis per datum quodvis punctum M, duci poteſt tangens MT; modò æquatio pro Parabola fuerit $y^n = x^m a^{n-m}$, & pro Hyperbola, $x^m y^n = a^{m+n}$; ſumatur ſcilicet ſubtangens $PT = \frac{n}{m}AP$, ex eadem parte quà punctum A reſpectu habito ad P, quando tangens ad Parabolam ſit ducenda; ex oppoſita autem parte, quando tangens ad Hyperbolam ſit ducenda;

PROPOSITIO XVI.

THEOREMA.

241. Sit, *ut in definitione quarta,* AMB *Parabola cucujuſcunque dimenſionis, per æquationem* $y^n = x^m a^{n-m}$ *expreſſa; ducatur autem in ea ex puncto quovis* B *recta* BC, *quæ cum* AC *angulum datum* ACB *conſtituat, compleaturque parallelogrammum* ACBD; *dico parallelogrammum circumſcriptum* ACBD *eſſe ad ſpatium Parabolicum* ACBMA *inter rectas* AC, CB *& Parabolæ portionem* AMB *comprehenſum, ſicut* $m + n$ *ad* n. Fig. 114.

Scilicet ACBD : ACBMA :: $m + v : n$.

Sumatur ſuper Parabolæ AMB portionem arcus MN infinitè vel indefinitè parvus, hoc eſt minor quavis Parabolæ portione, utut exigua: ducanturque rectæ MP, NQ, ipſi BC; & MK, NL ipſi AC parallelæ; quæ exiguum parallelogrammum MRNS conſtituent; agatur tangens MT diametro AC in puncto T occurrens, & per T ducta parallela CB ipſis MK, NL, in punctis F, G, occurret. Hoc poſito, conſtat arcum exiguum MN haberi * poſſe pro latere uno Polygoni Portionem Paraborabolæ AMB conſtituentis, tangentem vero pro latere illo producto; adeo ut duo triangula NRM, MPT, ſint & rectilinea & ſimilia, unde erit NR vel MS : RM :: MP : PT vel MF. Ergo erit Parallelogrammum PMRQ * æquale parallelogrammo FMSG; jam vero MF vel * $PT = \frac{n}{m}$ AP vel $\frac{n}{m}$ MK. Ergo etiam Parallelogrammum FMSG vel $PMRQ = \frac{n}{m}$ KMSL, & cum hoc idem fit

* Art. 189.

* 14. El. 6.

* Art. 240.

quacunque in Parabolæ parte cadat arcus exiguus MN, ſequitur ſummam omnium parallelogrammorum PMRQ, * Art. 184. hoc eſt, * Trilineum Parabolicum ACBMA $= \frac{n}{m}$ ADBMA ſummæ ſcilicet omnium parallelogrammorum $\frac{n}{m}$KMSL.

Unde mACBMA $= n$ ADBMA,

* 16. El. 6. Et * ADBMA : ACBMA :: $m : n$.

Ergo ADBMA + ACBMA : ACBMA :: $m + n : n$.

Hoc eſt, ACBD : ACBMA :: $m + n : n$.

COROLLARIUM I.

242. HINC conſtat Trilineum Parabolicum APM eſſe ad Parallelogrammum circumſcriptum APMK, ſicut n ad $m + n$; ideoque trapezium Parabolicum MPCB $= \frac{n}{m+n}$ ABCD $- \frac{n}{n+m}$ APMK; ex eo quod ACBMA $= \frac{n}{m+n}$ ACBD, & APM $= \frac{n}{n+m}$ APMK.

COROLLARIUM II.

243. SI Parabola data AMB fuerit Apolloniana, hoc eſt ſecundæ dimenſionis, æquatio generalis $y^n = x^m a^{n-m}$ evadit $y^2 = ax$; unde in hoc caſu $n = 2$, $m = 1$; ideoque

ACBD : ACBMA :: 3 : 2.

Si Parabola data eſſet Cubica, vel tertiæ demenſionis, æquatio generalis evaderet in hoc caſu $y^3 = ax^2$, ideoque

ACBD : ACBMA :: 5 : 3. Et ſic in infinitum.

PROPOSITIO XVII.

THEOREMA.

244. Sit, *ut in definitione quinta,* BMO *Hyperbola cujuſcunque dimenſionis per æquationem* $x^m y^n = a^{m+n}$ *deſignata; ducatur autem ex puncto quovis* C *recta* BC *alteri Aſymptoto* AD *parallela, & ad alteram in puncto* C *terminata, compleaturque parallelogrammum* ACBD; *dico parallelogrammum inſcriptum* ACBD *eſſe ad ſpatium Hyperbolicum* ECBMO *ſub recta determinata* BC, *ſub recta* CE *infinita verſus* C, *& ſub portione Hyperbolicâ* BMO *comprehenſum, ſicut* m — n : n. Fig. 115.

Scilicet ACBD : ECBMO :: $m - n : n$.

Manente præcedentis propoſitionis conſtructione, eadem ratione oſtendi poteſt, exiguum parallelogrammum PMRQ $= \frac{n}{m}$ KMSL. Et cum hoc verum ſit, quacunque in Hyperbolæ BMO parte cadat arcus MN, ſequitur ſummam omnium parallelogrammorum PMRQ, hoc eſt, * ſpatium ECBMO, æquari ſpatio $\frac{n}{m}$ EADBMO ſummæ videlicet parallelogrammorum omnium $\frac{n}{m}$ KMSL. Erit igitur

* Art. 184.

EADBMO : ECBMO :: $m : n$,
Et EADBMO — ECBMO : ECBMO :: $m - n : n$.
Hoc eſt, ACDB : ECBMO :: $m - n : n$.

COROLLARIUM I.

245. Hinc ſequitur trapezium hyperbolicum CPMB

=

$=\frac{n}{m-n}$ ACBD $=\frac{n}{m-n}$ APMK; ex eo quod ECBMO $=\frac{n}{m-n}$ ACBD, & quod ob eandem rationem EPMO $=\frac{n}{m-n}$ APMK.

COROLLARIUM II.

246. HINC, *1mo.* Si m major ſit quam n, ratio parallelogrammi inſcripti ACBD ad ſpatium ECBMO verſus E indefinitum, ſemper exprimitur per numeros poſitivos; unde in hoc caſu ſemper daretur abſoluta iſtiuſce ſpatii quadratura.

2do. QUANDO $m = n$, quod fit in Hyperbola communi vel Apolloniana, erit ACBD : ECBMO :: $o : 1$; hoc eſt ſpatium ECBMO infinitum eſt reſpectu habito ad parallelogrammum inſcriptum ABCD.

3tio. QUANDO m minor eſt quam n, parallelogrammum inſcriptum ACBD erit ad ſpatium hyperbolicum ECBMO, ut numerus negativus ad poſitivum; unde in hoc caſu ratio iſtiuſce ſpatii ad parallelogrammum ACBD eſt, ut ita dicam, plus quam infinita.

Notandum eſt tamen, quod in hoc poſtremo caſu ſpatium Hyperbolicum ſub recta DB, ſub Aſymptoto AD verſus D infinitâ, & ſub Hyperbola OMB comprehenſum, erit ad parallelogrammum inſcriptum ACBD ſicut m ad $n - m$, hoc eſt, ſpatii illius quadratura datur; ſi enim indeterminatæ (x) ſuper Aſymptoton AD jam ſumantur (ſumptæ enim erant antea ſuper AC) æquatio *Art. 233. ad Hyperbolam evadet * $x^n y^m = a^{m+n}$.

PRO-

PROPOSITIO XVIII.

THEOREMA.

247. *Si ſit in angulo recto* CAD *quævis curva* AMB, *cujus tangens* MT *in quovis puncto* M *datur; & ſi in angulo* DAH *qui deinceps ſit, conſtituatur alia curva* HFE *ejuſmodi; ut, ſi ſuper eam ex puncto quovis* F *ducatur recta* FM *ipſi* AC *parallela, rectæ* AD *in* K, *curvæ autem in* M *occurrens, agaturque tangens* MT *ipſi* AC *in puncto* T *occurrens: ſit ſemper* AK *ad* MT *ſicut conſtans quædam* a *(eadem ſemper manens, quacunque in parte cadat punctum* F*) ad* KF*; dico, ſi per punctum quodvis* D *in recta* AD *ducatur recta* EB *ipſi* AC *parallela & ad duas curvas terminata, ſpatium* ADEFH *æquari rectangulo ſub curva* AMB *& conſtanti* a *comprehenſo.*

Fig. 116.

Scilicet $ADEFH = AMB \times a$.

Sumpto ſuper curvam AMB arcu MN infinitè parvo, ductiſque MF, NG ipſi AC parallelis, rectæ AD in punctis K, L, curvæ autem HFE in punctis F & G occurrentibus, agantur rectæ FS, MR ipſi AD parallelæ, producaturque RM, uſque dum ipſi AC in P occurrat.

Hoc poſito triangula MPT, MRN, ſunt ſimilia, unde

$MR : MN :: MP$ vel $AK : MT :: a : KF$.

Ergo $KF \times MR$, hoc eſt, $FKLS = MN \times a$.

Et cum hoc verum ſit, quacunque in parte curvæ AMB ſumatur arcus MN, ſequitur ſummam exiguorum rectangulorum FKLS, hoc eſt, * ſpatium ADEFH æquari ſummæ exiguorum rectangulorum $MN \times a$, hoc eſt rectangulo ſub curva AMB & conſtanti a comprehenſo.

* Art. 184.

COR-

COROLLARIUM I.

248. HINC ſequitur rectangulum ſub portione AM & conſtanti a, æquari ſpatio AKFH; ita & rectangulum ſub portione MB & eadem recta a comprehenſum æquari ſpatio KDEF.

COROLLARIUM II.

249. SI curva AMB fuerit ſecunda Parabola cubica, * Art. 236. cujus æquatio ſit $y^3 = axx$ ($AP = x$, $PM = y$;) erit * $PT = \frac{3}{2}x$. Et ob triangulum rectangulum MPT, Hypothenuſa $MT = \sqrt{yy + \frac{9}{4}xx}$; verum ex natura curvæ HFE, erit $MP : MT :: a : KF$, vel $y : \sqrt{yy + \frac{9}{4}xx} :: a : KF$, Unde $\overline{KF}^2 = aa + \frac{9aaxx}{4yy}$; ſed $y^3 = axx$, ergo ſubſtitutione facta erit $\overline{KF}^2 = aa + \frac{9}{4}ay$. Hinc conſtat, curvam HFE eſſe in hoc caſu Parabolam, cujus axis eſt recta AD, vertex autem O, ita ut punctum D ſit ex ex altera parte puncti A, & O ex altera, adeo ut $AO = \frac{4}{9}a$ * Art. 22. & parameter ejus $= \frac{9}{4}a$; eſt enim * ex natura Parabolæ quadratum ipſius KF æquale rectangulo ſub KO & parametro $\frac{9}{4}a$: hoc eſt, $\overline{KF}^2 = aa + \frac{9}{4}ay$. Jam vero cum * Art. 242. trapezia Parabolica ADEH, AKFH * quadrari poſſint, conſtat dari rectificationem & curvæ AMB, & portionis cujuſvis AM.

Si quæratur vera quantitas portionis AM, notandum eſt $AH = a$, quoniam $\overline{AH}^2 = AO \times \frac{9}{4}a = aa$; adeoque facta tangente $MT = t$, recta AK vel $MP = y$,

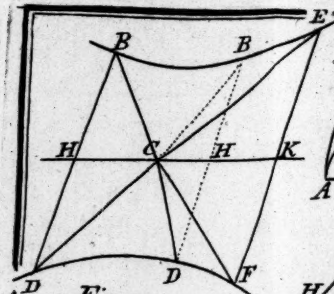

Fig:104

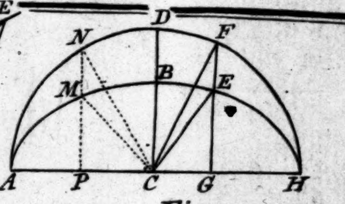

Fig:105.

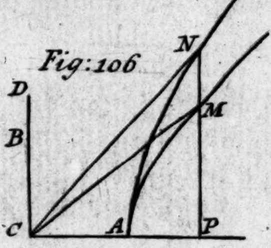

Fig:106

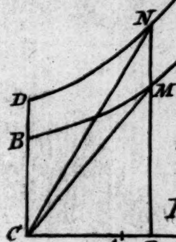

Fig:107

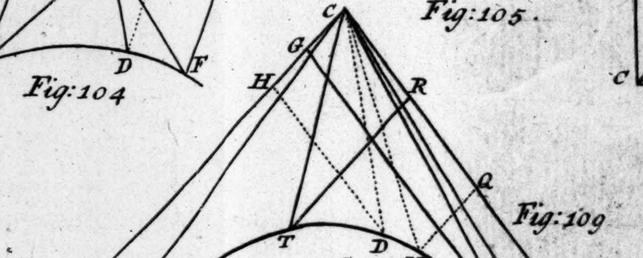

Fig:109

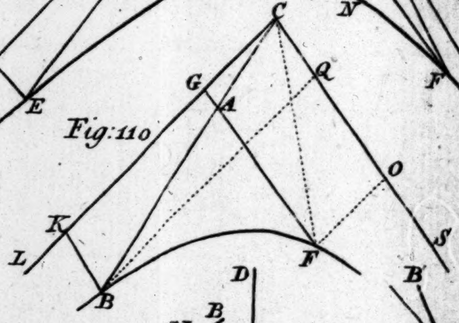

Fig:110

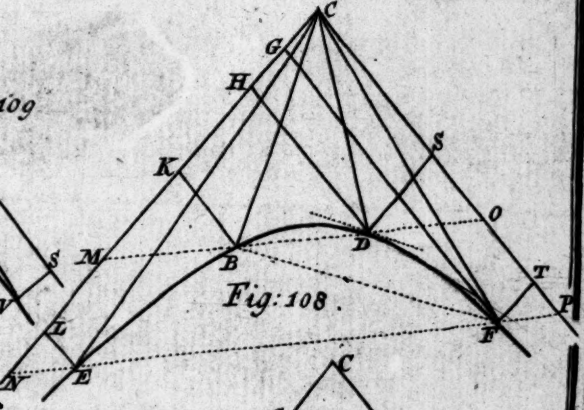

Fig:108.

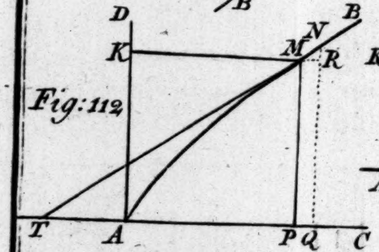

Fig:112

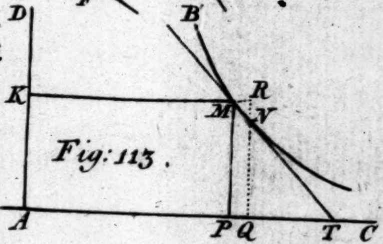

Fig:113.

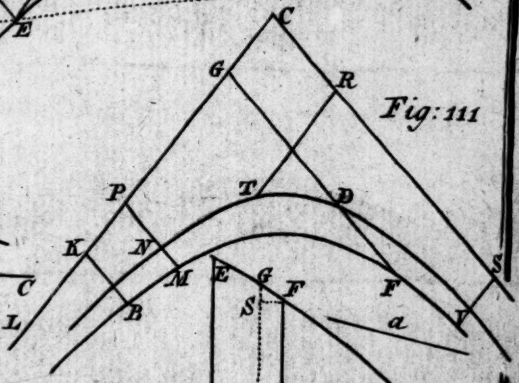

Fig:111

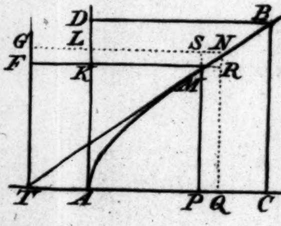

Fig:114.

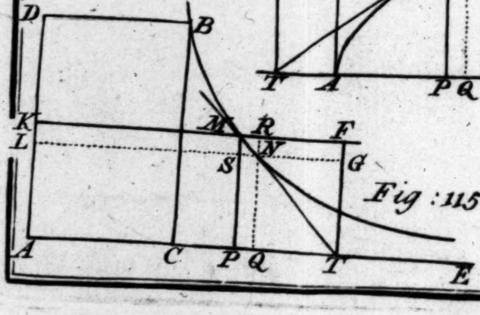

Fig:115.

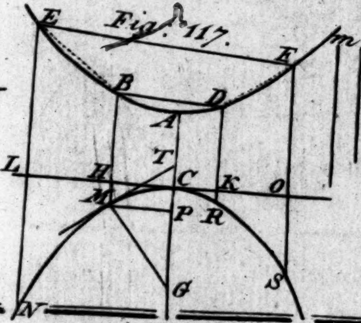

Fig:117.

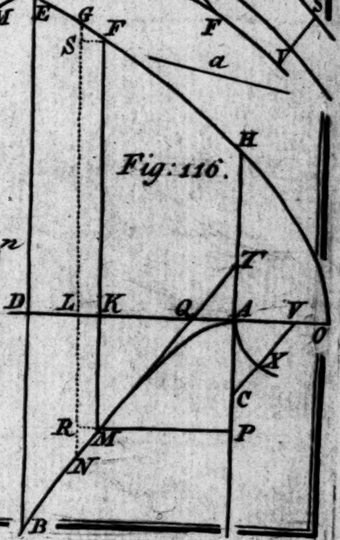

Fig:116.

erit KF $=\frac{at}{y}$, & trapezium Parabolicum FKAH vel * $\frac{2}{3}$ FK × KO $- \frac{2}{3}$ HA × AO $= \frac{2}{3} at + \frac{8aat}{27y} - \frac{8}{27} aa =$ AM × a. Hoc eſt, dividendo per a, portio AM $= \frac{2}{3} t + \frac{8at}{27y} - \frac{8}{27} a$. Unde oritur hæc conſtructio.

* Art. 242.

Actâ ex puncto dato M ſuper ſuper ſecundam Parabolam cubicam AMB tangente MT, quæ ipſi AK per axis AC verticem ad eundem axem perpendiculariter ductæ normalis ſit, ſumatur ſuper rectam AK pars AV $= \frac{8}{27} a$, ducaturque VC ipſi MT parallela axi in puncto C occurrens; deſcribatur centro N radio autem VA arcus Circularis AX ipſam VC in puncto X ſecans. Dico portionem AM ſecundæ Parabolæ cubicæ AMB æquari ſummæ rectarum MQ, CX.

Ob triangula enim TPM, TAQ ſimilia, erit MQ $= \frac{2}{3}$ MT (t), ex eo quod AP $= \frac{2}{3}$ PT; ob triangula ſimilia MPT, VAC, erit

$$\text{MP} : \text{MT} :: \text{AV} : \text{VC},$$

Hoc eſt, $y : t :: \frac{8}{27} a : \frac{8at}{27y}$, unde CX $= \frac{8at}{27y} - \frac{8}{27} a$; ſed MQ $= \frac{2}{3}$ MT, ideoque MQ + CX $= \frac{2}{3} t + \frac{8at}{27y} - \frac{8}{27} a =$ AM.

PROPOSITIO XIX.

THEOREMA.

Fig. 117. 250. SIT EAF *Hyperbola Equilatera, cujus centrum* C, *ſemiaxis primus* AC; *ſit etiam* NCS *Parabola, cujus axis fuerit recta* AC *verſus Verticem* C *producta, axis autem Parameter dupla* CA. *Jam ſi per quodvis in Parabola* NCS *punctum* N, *agatur recta* NE *ipſi* CA *parallela, Hyperbolæ ipſi* EAF *in puncto* E, *ejus autem axi ſecundo* CL *in puncto* L *occurrens; dico ſpatium Hyperbolicum* CLEA *ſub rectis* AC, CL, LE, *& ſub Hyperbolæ portione* EA *comprehenſum, æquari rectangulo ſub Parabolæ portione* CN, *& recta* AC *contento.*

Scilicet CLEA $=$ CN $\times$ AC.

Ducatur per punctum quodvis M in portione Parabolica CN recta MG tangenti MT per illud punctum ductæ, & ad axem in T terminatæ perpendicularis, ad axem etiam in G terminata; agatur MB ipſi CA parallela, Hyperbolæ in puncto B, ſecundo autem axi CL in puncto H occurrens; & erunt rectæ MG, HB, ſibi invicem æquales. Ducta enim ad axem Ordinata MP,
* Art. 28. erit * PG $=$ CA; & ob triangulum rectangulum MPG,
* Art. 140. erit etiam $\overline{MG}^2 = \overline{PM}^2 + \overline{PG}^2 = \overline{CH}^2 + \overline{CA}^2 =$ *$\overline{HB}^2$, ob Hyperbolam equilateram EAF; unde MG $=$ HB. Triangula autem TPM, MPG ſunt ſimilia, unde MP vel
* Art. 246. CH : MT :: PG vel CA : MG vel HB. Ergo * ſpatium Hyperbolicum CLEA $=$ CN $\times$ AC.

COROLLARIUM II.

251. HINC conſtat, trapezium Hyperbolicum HLEB

æquari rectangulo ſub portione Parabolica MN, & ſemiparametro CA axis.

COROLLARIUM II.

252. Si in Hyperbola equilatera EAF agantur duæ parallelæ BD, EF, ducanturque per earum extremitates rectæ BM, EN, DR, FS, ipſi AC parallelæ, ſecundo axi Hyperbolæ in punctis H, K, L, O, occurrentes, differentia rectangulorum $AC \times MN$, $AC \times RS$ æquabitur differentiæ trapeziorum rectilineorum HLEB, KOFD.

Nam rectangulum $AC \times MN$ *æquatur trapezio Hyperbolico HLEB, ideoque rectangulum $AC \times MN +$ ſegmentnm Hyperbolicum $EB =$ trapezio rectilineo HLEB; eadem ratione rectangulum $AC \times RS +$ ſegmentum Hyperbolicum $DF =$ trapezio rectilineo KOFD; ſed ſegmenta EB, DF *æquantur; ergo differentia rectangulorum $AC \times MN$, $AC \times RS$ æquabitur differentiæ trapeziorum rectilineorum HLEB, KOFD.

* Art. 250.

* Art. 208.

COROLLARIUM III.

253. IISDEM, quæ in conſectario præcedenti, poſitis; ſi fiat $2AC : LH :: BH + LE : m$; liquet rectangulum $AC \times m = \frac{1}{2} LH \times \overline{BH + LE}$, hoc eſt trapezio rectilineo HLEB; eadem ratione, ſi fiat $2AC : KO :: KD + FO : n$, erit $AC \times n =$ trapezio KOFD; ideoque differentia rectangulorum $AC \times RS$, æquabitur differentiæ rectangulorum $AC \times m$, $AC \times n$; hoc eſt, dividendo per AC, differentia arcuum Parabolicorum MN, RS, æquabitur differentiæ rectarum m, n; unde conſtat inveniri poſſe rectas æquales differentiæ arcuum Parabolicorum MN, RS, ad infinitum.

FINIS.

MVSEVM
BRITAN
NICVM

CORRIGENDA.

PAG l. 6. lege *dimidio.* p. 40 l. 16. lege $\frac{2abyy}{sx}$ pro $\frac{2abyy}{xx}$ p. 70. l. ult. leg. *æquari.* p. 88. l. ult. pro *Ch* lege CH. p. 101. l. 2. leg. *illam.* p. 151. l. 18. leg. *numerorum.* p. 141, l. 7. pro ZPY leg. SPZ. p. 143. l. 1. lege *Diametro.*